

Chapter 16

Complex Analysis

The term “complex analysis” refers to the calculus of complex-valued functions $f(z)$ depending on a single complex variable z . On the surface, it may seem that this subject should merely be a simple reworking of standard real variable theory that you learned in first year calculus. However, this naïve first impression could not be further from the truth! Complex analysis is the culmination of a deep and far-ranging study of the fundamental notions of complex differentiation and complex integration, and has an elegance and beauty not found in the more familiar real arena. For instance, complex functions are always *analytic*, meaning that they can be represented as convergent power series. As an immediate consequence, a complex function automatically has an *infinite* number of derivatives, and difficulties with degree of smoothness, strange discontinuities, delta functions, and other forms of pathological behavior of real functions never arise in the complex realm.

The driving force behind many applications of complex analysis is the remarkable and profound connection between harmonic functions (solutions of the Laplace equation) of two variables and complex functions. Namely, the real and imaginary parts of a complex analytic function are automatically harmonic. In this manner, complex functions provide a rich lode of new solutions to the two-dimensional Laplace equation to help solve boundary value problems. One of the most useful practical consequences arises from the elementary observation that the composition of two complex functions is also a complex function. We interpret this operation as a complex change of variables, also known as a *conformal mapping* since it preserves angles. Conformal mappings can be effectively used for constructing solutions to the Laplace equation on complicated planar domains, and play a particularly important role in the solution of physical problems.

Complex integration also enjoys many remarkable properties not found in its real sibling. Integrals of complex functions are similar to the line integrals of planar multivariable calculus. The remarkable theorem due to Cauchy implies that complex integrals are generally path-independent — provided one pays proper attention to the complex singularities of the integrand. In particular, an integral of a complex function around a closed curve can be directly evaluated through the “calculus of residues”, which effectively bypasses the Fundamental Theorem of Calculus. Surprisingly, the method of residues can even be applied to evaluate certain types of definite real integrals.

In this chapter, we shall introduce the basic techniques and theorems in complex analysis, paying particular attention to those aspects which are required to solve boundary value problems associated with the planar Laplace and Poisson equations. Complex analysis is an essential tool in a surprisingly broad range of applications, including fluid flow,

elasticity, thermostatics, electrostatics, and, in mathematics, geometry, and even number theory. Indeed, the most famous unsolved problem in all of mathematics, the Riemann hypothesis, is a conjecture about a specific complex function that has profound consequences for the distribution of prime numbers[†].

16.1. Complex Variables.

In this section we shall develop the basics of complex analysis — the calculus of complex functions $f(z)$. Here $z = x + iy$ is a single complex variable and $f: \Omega \rightarrow \mathbb{C}$ is a complex-valued function defined on a domain $z \in \Omega \subset \mathbb{C}$ in the complex plane. Before diving into this material, you should first make sure you are familiar with the basics of complex numbers, as discussed in Section 3.6.

Any complex function can be written as a complex combination

$$f(z) = f(x + iy) = u(x, y) + iv(x, y), \quad (16.1)$$

of two real functions u, v depending on two real variables x, y , called, respectively, its *real* and *imaginary parts*, and written

$$u(x, y) = \operatorname{Re} f(z), \quad \text{and} \quad v(x, y) = \operatorname{Im} f(z). \quad (16.2)$$

For example, the monomial function $f(z) = z^3$ is written as

$$z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3),$$

and so

$$\operatorname{Re} z^3 = x^3 - 3xy^2, \quad \operatorname{Im} z^3 = 3x^2y - y^3.$$

We can identify $\mathbb{C}$ with the real, two-dimensional plane $\mathbb{R}^2$, so that the complex number $z = x + iy \in \mathbb{C}$ is identified with the real vector $(x, y)^T \in \mathbb{R}^2$. Based on this identification, we shall employ the standard terminology of planar vector calculus, e.g., domain, curve, etc., without alteration; see Appendix A for details. In this manner, we may regard a complex function as particular type of real vector field that maps

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \Omega \subset \mathbb{R}^2 \quad \text{to the vector} \quad \mathbf{v}(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \in \mathbb{R}^2. \quad (16.3)$$

Not every real vector field qualifies as a complex function; the components $u(x, y), v(x, y)$ must satisfy certain fairly stringent requirements, which can be found in Theorem 16.3 below.

Many of the well-known functions appearing in real-variable calculus — polynomials, rational functions, exponentials, trigonometric functions, logarithms, and many others — have natural complex extensions. For example, complex polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad (16.4)$$

[†] Not to mention that a solution will net you a cool \$1,000,000.00. For details on how to claim your prize, check out the web site <http://www.claymath.org>.

are complex linear combinations (meaning that the coefficients a_k are allowed to be complex numbers) of the basic monomial functions $z^k = (x + i y)^k$. Similarly, we have already made sporadic use of complex exponentials such as

$$e^z = e^{x+iy} = e^x \cos y + i e^x \sin y$$

for solving differential equations. Other examples will appear shortly.

There are several ways to motivate[†] the link between harmonic functions $u(x, y)$, meaning solutions of the two-dimensional Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (16.5)$$

and complex functions. One natural starting point is to return to the d'Alembert solution (14.124) of the one-dimensional wave equation, which was based on the factorization

$$\square = \partial_t^2 - c^2 \partial_x^2 = (\partial_t - c \partial_x)(\partial_t + c \partial_x)$$

of the linear wave operator (14.113). The two-dimensional Laplace operator $\Delta = \partial_x^2 + \partial_y^2$ has essentially the same form, except for a “minor” change in sign[‡]. We cannot produce a real factorization of the Laplace operator, but there is a complex factorization,

$$\Delta = \partial_x^2 + \partial_y^2 = (\partial_x - i \partial_y)(\partial_x + i \partial_y),$$

into a product of two complex first order differential operators, having complex “wave speed” $c = i$. Mimicking the solution formula (14.121) for the wave equation, we expect that the solutions to the Laplace equation (16.5) should be expressed in the form

$$u(x, y) = f(x + i y) + g(x - i y), \quad (16.6)$$

i.e., a linear combination of functions of the complex variable $z = x + i y$ and its complex conjugate $\bar{z} = x - i y$. The functions $f(x + i y)$ and $g(x - i y)$ satisfy the first order complex partial differential equations

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}, \quad \frac{\partial g}{\partial x} = i \frac{\partial g}{\partial y}, \quad (16.7)$$

and hence (16.6) does indeed define a complex-valued solution to the Laplace equation.

In most applications, we are searching for a real solution to the Laplace equation, and so our d'Alembert-type formula (16.6) is not entirely satisfactory. As we know, a complex number $z = x + i y$ is real if and only if it equals its own conjugate, $z = \bar{z}$. Thus, the solution (16.6) will be real if and only if

$$f(x + i y) + g(x - i y) = u(x, y) = \overline{u(x, y)} = \overline{f(x + i y) + g(x - i y)}.$$

[†] A reader uninterested in the motivation can skip ahead to Proposition 16.1 at this point.

[‡] However, the change in sign has serious ramifications for the analytical properties of solutions to the two equations. As noted in Section 15.1, there is a profound difference between the elliptic Laplace equation and the hyperbolic wave equation.

Now, the complex conjugation operation switches $x + iy$ and $x - iy$, and so we expect the first term $\overline{f(x + iy)}$ to be a function of $x - iy$, while the second term $\overline{g(x - iy)}$ will be a function of $x + iy$. Therefore[§], to equate the two sides of this equation, we should require

$$g(x - iy) = \overline{f(x + iy)},$$

and so

$$u(x, y) = f(x + iy) + \overline{f(x + iy)} = 2 \operatorname{Re} f(x + iy).$$

Dropping the inessential factor of 2, we conclude that a real solution to the two-dimensional Laplace equation can be written as the *real part* of a complex function. A direct proof of the following key result will appear below.

Proposition 16.1. *If $f(z)$ is a complex function, then its real part*

$$u(x, y) = \operatorname{Re} f(x + iy) \tag{16.8}$$

is a harmonic function.

The *imaginary part* of a complex function is also harmonic. This is because

$$\operatorname{Im} f(z) = \operatorname{Re} (-i f(z))$$

is the real part of the complex function

$$-i f(z) = -i[u(x, y) + iv(x, y)] = v(x, y) - iu(x, y).$$

Therefore, if $f(z)$ is any complex function, we can write it as a complex combination

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

of two real harmonic functions: $u(x, y) = \operatorname{Re} f(z)$ and $v(x, y) = \operatorname{Im} f(z)$.

Before delving into the many remarkable properties of complex functions, let us look at some of the most basic examples. In each case, the reader can directly check that the harmonic functions given as the real and imaginary parts of the complex function are indeed solutions to the Laplace equation.

Examples of Complex Functions

(a) *Harmonic Polynomials:* The simplest examples of complex functions are polynomials. Any polynomial is a complex linear combinations, as in (16.4), of the basic complex monomials

$$z^n = (x + iy)^n = u_n(x, y) + iv_n(x, y). \tag{16.9}$$

The real and imaginary parts of a complex polynomial are known as *harmonic polynomials*, and we list the first few below. The general formula for the basic harmonic polynomials

[§] We are ignoring the fact that f and g are not quite uniquely determined since one can add and subtract a constant from them. This does not affect the argument in any significant way.

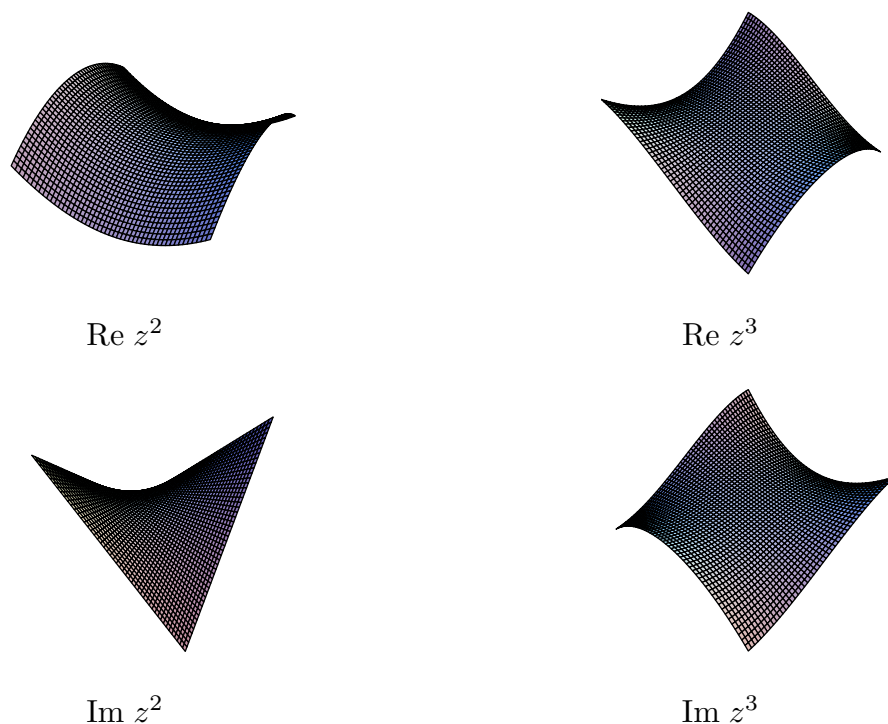


Figure 16.1. Real and Imaginary Parts of z^2 and z^3 .

$u_n(x, y)$ and $v_n(x, y)$ is easily found by applying the binomial theorem to expand (16.9), as in Exercise ■.

Harmonic Polynomials

n	z^n	$u_n(x, y)$	$v_n(x, y)$
0	1	1	0
1	$x + iy$	x	y
2	$(x^2 - y^2) + 2ixy$	$x^2 - y^2$	$2xy$
3	$(x^3 - 3xy^2) + i(3x^2y - y^3)$	$x^3 - 3xy^2$	$3x^2y - y^3$
4	$(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$	$x^4 - 6x^2y^2 + y^4$	$4x^3y - 4xy^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We have, in fact, already encountered these polynomial solutions to the Laplace equation. If we write

$$z = r e^{i\theta}, \quad (16.10)$$

where

$$r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \text{ph } z = \tan^{-1} \frac{y}{x},$$

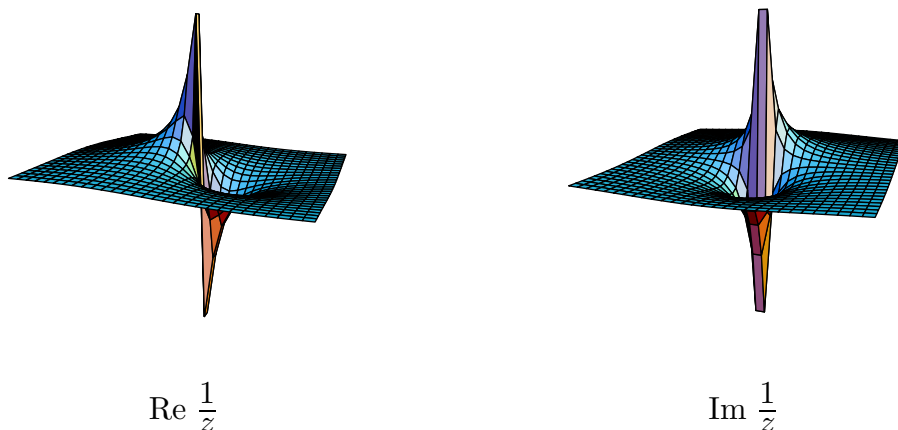


Figure 16.2. Real and Imaginary Parts of $f(z) = \frac{1}{z}$.

are the usual polar coordinates (modulus and phase) of $z = x + iy$, then Euler's formula (3.84) yields

$$z^n = r^n e^{in\theta} = r^n \cos n\theta + i r^n \sin n\theta,$$

and so

$$u_n = r^n \cos n\theta, \quad v_n = r^n \sin n\theta.$$

Therefore, the harmonic polynomials are just the polar coordinate separable solutions (15.38) to the Laplace equation. In Figure 16.1 we plot[†] the real and imaginary parts of the monomials z^2 and z^3 .

(b) *Rational Functions*: Ratios

$$f(z) = \frac{p(z)}{q(z)} \tag{16.11}$$

of complex polynomials provide a large variety of harmonic functions. The simplest case is

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}. \tag{16.12}$$

Its real and imaginary parts are graphed in Figure 16.2. Note that these functions have an interesting singularity at the origin $x = y = 0$, but are harmonic everywhere else.

A slightly more complicated example is the function

$$f(z) = \frac{z - 1}{z + 1}. \tag{16.13}$$

[†] Graphing a complex function $f: \mathbb{C} \rightarrow \mathbb{C}$ is problematic. The identification (16.3) of f with a real vector-valued function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ implies that four real dimensions are needed to display its complete graph.

To write out (16.13) in real form, we multiply and divide by the complex conjugate of the denominator, leading to

$$f(z) = \frac{z-1}{z+1} = \frac{(z-1)(\bar{z}+1)}{(z+1)(\bar{z}+1)} = \frac{|z|^2 + z - \bar{z} - 1}{|z+1|^2} = \frac{x^2 + y^2 - 1}{(x+1)^2 + y^2} + i \frac{2y}{(x+1)^2 + y^2}. \quad (16.14)$$

This manipulation can always be used to find the real and imaginary parts of general rational functions.

(c) *Complex Exponentials*: Euler's formula

$$e^z = e^x \cos y + i e^x \sin y \quad (16.15)$$

for the complex exponential, cf. (3.84), yields two important harmonic functions: $e^x \cos y$ and $e^x \sin y$, which are graphed in Figure 3.8. More generally, writing out e^{cz} for a complex constant $c = a + ib$ produces the complex exponential function

$$e^{cz} = e^{ax-by} \cos(bx+ay) + i e^{ax-by} \sin(bx+ay). \quad (16.16)$$

Its real and imaginary parts are harmonic functions for arbitrary $a, b \in \mathbb{R}$. Some of these were found by applying the separation of variables method in Cartesian coordinates; see the table in Section 15.2.

(d) *Complex Trigonometric Functions*: The complex trigonometric functions are defined in terms of the complex exponential by adapting our earlier formulae (3.86):

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} = \cos x \cosh y - i \sin x \sinh y, \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} = \sin x \cosh y + i \cos x \sinh y. \end{aligned} \quad (16.17)$$

The resulting harmonic functions are products of trigonometric and hyperbolic functions. They can all be written as linear combinations of the harmonic functions (16.16) derived from the complex exponential. Note that when $z = x$ is real, so $y = 0$, these functions reduce to the usual real trigonometric functions $\cos x$ and $\sin x$.

(e) *Complex Logarithm*: In a similar fashion, the complex logarithm $\log z$ is a complex extension of the usual real natural (i.e., base e) logarithm. In terms of polar coordinates (16.10), the complex logarithm has the form

$$\log z = \log(r e^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta, \quad (16.18)$$

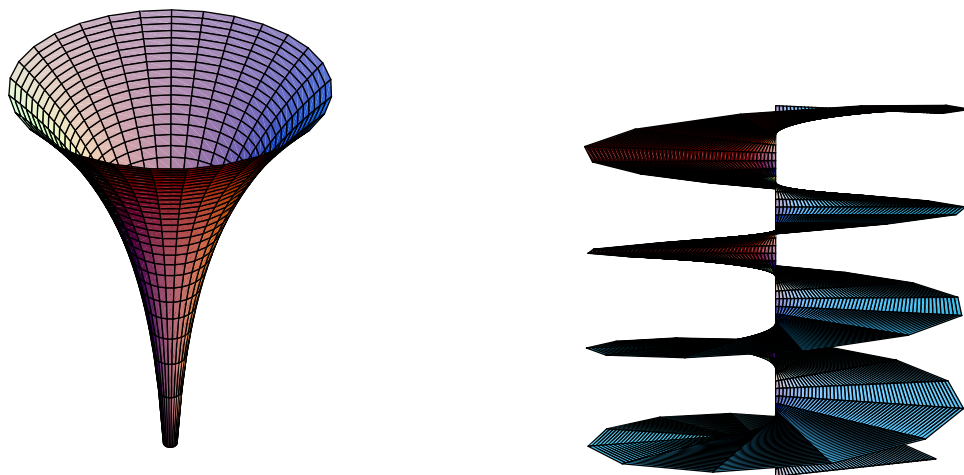
Thus, the logarithm of a complex number has real part

$$\operatorname{Re}(\log z) = \log r = \log |z| = \frac{1}{2} \log(x^2 + y^2),$$

which is a well-defined harmonic function on all of $\mathbb{R}^2$ except for a logarithmic singularity at the origin $x = y = 0$. It is, in fact, the logarithmic potential corresponding to a delta function forcing concentrated at the origin that played a key role in our construction of the Green's function for the Poisson equation in Section 15.3.

The imaginary part

$$\operatorname{Im}(\log z) = \theta = \operatorname{ph} z$$



$$\operatorname{Re}(\log z) = \log |z|$$

$$\operatorname{Im}(\log z) = \operatorname{ph} z$$

Figure 16.3. Real and Imaginary Parts of $\log z$.

of the complex logarithm is the *phase* (argument) or polar angle of z . The phase is also not defined at the origin $x = y = 0$. Moreover, it is a multi-valued harmonic function elsewhere, since it is only specified up to integer multiples of 2π . Thus, a given nonzero complex number $z \neq 0$ has an infinite number of possible values for its phase, and hence an infinite number of possible complex logarithms $\log z$, each differing by an integer multiple of $2\pi i$, reflecting the fact that $e^{2\pi i} = 1$. In particular, if $z = x > 0$ is real and positive, then $\log z = \log x$ agrees with the real logarithm, *provided* we choose the angle $\operatorname{ph} z = 0$. Alternative choices for the phase include an integer multiple of $2\pi i$, and so ordinary real, positive numbers $x > 0$ also have complex logarithms! On the other hand, if $z = x < 0$ is real and negative, then $\log z = \log |x| + (2k + 1)\pi i$ is complex no matter which value of $\operatorname{ph} z$ is chosen. (This explains why we didn't attempt to define the logarithm of a negative number in first year calculus!) As the point z circles around the origin in a counter-clockwise direction, $\operatorname{Im} \log z = \operatorname{ph} z = \theta$ increases by 2π . Thus, its graph can be likened to a parking ramp with infinitely many levels, spiraling ever upwards as one circumambulates the origin; Figure 16.3 attempts to sketch it. For the complex logarithm, the origin is a type of singularity known as a *logarithmic branch point*, the “branches” referring to the infinite number of possible values that can be assigned to $\log z$ at any nonzero point.

(f) *Roots and Fractional Powers:* A similar branching phenomenon occurs with the fractional powers and roots of complex numbers. The simplest case is the square root function $\sqrt{z}$. Every nonzero complex number $z \neq 0$ has two different possible square roots: $\sqrt{z}$ and $-\sqrt{z}$. As illustrated in Figure 16.4, the two square roots lie on opposite sides of the origin, and are obtained by multiplying by -1 . Writing $z = r e^{i\theta}$ in polar coordinates, we see that

$$\sqrt{z} = \sqrt{r e^{i\theta}} = \sqrt{r} e^{i\theta/2} = \sqrt{r} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right), \quad (16.19)$$

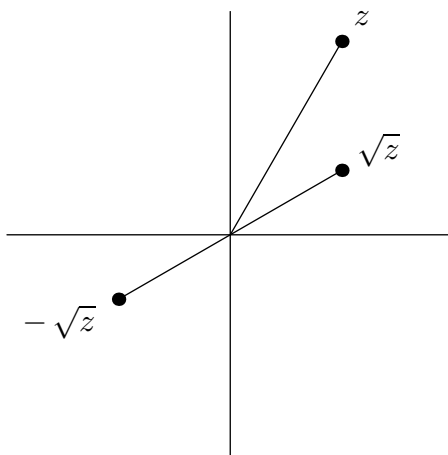


Figure 16.4. Square Roots of a Complex Number.

i.e., we take the square root of the modulus and halve the phase:

$$|\sqrt{z}| = \sqrt{|z|} = \sqrt{r}, \quad \text{ph } \sqrt{z} = \frac{1}{2} \text{ph } z = \frac{1}{2} \theta.$$

Since $\theta = \text{ph } z$ is only defined up to an integer multiple of 2π , the angle $\frac{1}{2}\theta$ is only defined up to an integer multiple of π . The even and odd multiples yield different values for (16.19), which accounts for the two possible values of the square root. For instance, since $\text{ph } 4i = \frac{1}{2}\pi$ or $\frac{5}{2}\pi$, we find

$$\sqrt{4i} = 2\sqrt{i} = \pm 2e^{\pi i/4} = \pm 2 \left(\cos \frac{\pi i}{4} + i \sin \frac{\pi i}{4} \right) = \pm (\sqrt{2} + i\sqrt{2}).$$

If we start at some $z \neq 0$ and circle once around the origin, we increase $\text{ph } z$ by 2π , but $\text{ph } \sqrt{z}$ only increases by π . Thus, at the end of our circuit, we arrive at the other square root $-\sqrt{z}$. Circling the origin again increases $\text{ph } z$ by a further 2π , and hence brings us back to the original square root $\sqrt{z}$. Therefore, the graph of the multiply-valued square root function will look like a weirdly interconnected parking ramp with only two levels, as shown in[†] Figure 16.5.

Similar remarks apply to the n^{th} root

$$\sqrt[n]{z} = \sqrt[n]{r} e^{i\theta/n} = \sqrt[n]{r} \left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right), \quad (16.20)$$

which, except for $z = 0$, has n possible values, depending upon which multiple of 2π is used in the assignment of $\text{ph } z = \theta$. The n different n^{th} roots are obtained by multiplying any one of them by the different n^{th} roots of unity, $\zeta_n^k = e^{2k\pi i/n}$ for $k = 0, \dots, n-1$, as defined in (13.11). In this case, the origin $z = 0$ is called a *branch point* of *order* n since there are n different branches for the function $\sqrt[n]{z}$. Circling around the origin a total of n times leads to the n branches in succession, returning in the end to the original.

[†] These graphs are best appreciated in an interactive three-dimensional graphics viewer.

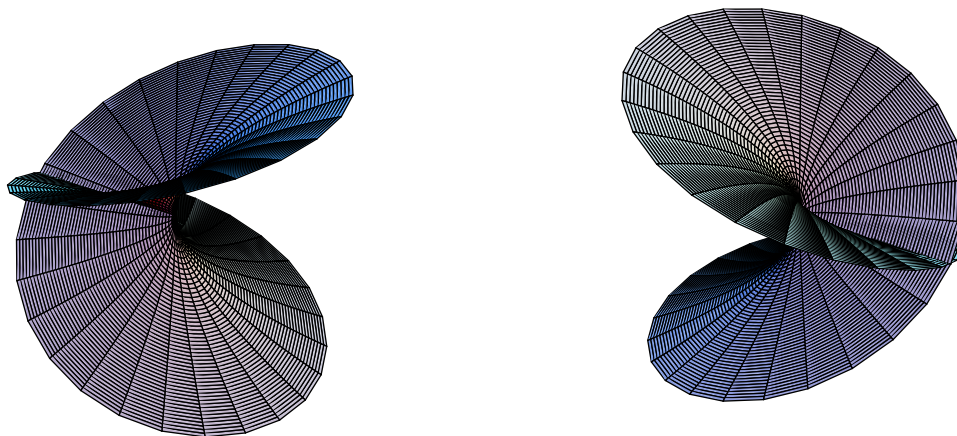


Figure 16.5. Real and Imaginary Parts of $\sqrt{z}$.

The preceding list of elementary examples is far from exhausting the range and variety of complex functions. Lack of space will preclude us from studying the remarkable properties of complex versions of the gamma function, Airy functions, Bessel functions, and Legendre functions that appear in Appendix C, as well as elliptic functions, the Riemann zeta function, modular functions, and many, many other important and fascinating functions arising in complex analysis and its manifold applications; see [175, 186].

16.2. Complex Differentiation.

The bedrock of complex function theory is the notion of the complex derivative. Complex differentiation is defined in the same manner as the usual calculus limit definition of the derivative of a real function. Yet, despite a superficial similarity, complex differentiation is profoundly different, and displays an elegance and depth not shared by its real progenitor.

Definition 16.2. A complex function $f(z)$ is *differentiable* at a point $z \in \mathbb{C}$ if and only if the limiting difference quotient exists:

$$f'(z) = \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}. \quad (16.21)$$

The key feature of this definition is that the limiting value $f'(z)$ of the difference quotient must be *independent* of how w converges to z . On the real line, there are only two directions to approach a limiting point — either from the left or from the right. These lead to the concepts of left and right handed derivatives and their equality is required for the existence of the usual derivative of a real function. In the complex plane, there are an infinite variety of directions to approach the point z , and the definition requires that all of these “directional derivatives” must agree. This is the reason for the more severe

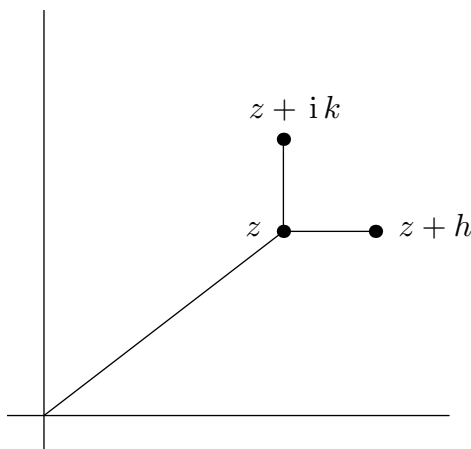


Figure 16.6. Complex Derivative Directions.

restrictions on complex derivatives, and, in consequence, the source of their remarkable properties.

Let us first see what happens when we approach z along the two simplest directions — horizontal and vertical. If we set

$$w = z + h = (x + h) + i y, \quad \text{where } h \text{ is real,}$$

then $w \rightarrow z$ along a horizontal line as $h \rightarrow 0$, as sketched in Figure 16.6. If we write out

$$f(z) = u(x, y) + i v(x, y)$$

in terms of its real and imaginary parts, then we must have

$$\begin{aligned} f'(z) &= \lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f(x + h + i y) - f(x + i y)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x + h, y) - u(x, y)}{h} + i \frac{v(x + h, y) - v(x, y)}{h} \right] = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}, \end{aligned}$$

which follows from the usual definition of the (real) partial derivative. On the other hand, if we set

$$w = z + i k = x + i(y + k), \quad \text{where } k \text{ is real,}$$

then $w \rightarrow z$ along a vertical line as $k \rightarrow 0$. Therefore, we must also have

$$\begin{aligned} f'(z) &= \lim_{k \rightarrow 0} \frac{f(z + i k) - f(z)}{i k} = \lim_{k \rightarrow 0} \left[-i \frac{f(x + i(y + k)) - f(x + i y)}{k} \right] \\ &= \lim_{k \rightarrow 0} \left[\frac{v(x, y + k) - v(x, y)}{k} - i \frac{u(x, y + k) - u(x, y)}{k} \right] = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = -i \frac{\partial f}{\partial y}. \end{aligned}$$

When we equate the real and imaginary parts of these two distinct formulae for the complex derivative $f'(z)$, we discover that the real and imaginary components of $f(z)$ must satisfy a

certain homogeneous linear system of partial differential equations, named after Augustin–Louis Cauchy and Bernhard Riemann[†], two of the principal founders of modern complex analysis.

Theorem 16.3. *A function $f(z) = u(x, y) + i v(x, y)$, where $z = x + i y$, has a complex derivative $f'(z)$ if and only if its real and imaginary parts are continuously differentiable and satisfy the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (16.22)$$

In this case, the complex derivative of $f(z)$ is equal to any of the following expressions:

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \quad (16.23)$$

The proof of the converse — that any function whose real and imaginary components satisfy the Cauchy–Riemann equations is differentiable — will be omitted, but can be found in any basic text on complex analysis, e.g., [4, 156].

Remark: It is worth pointing out that equation (16.23) tells us that f satisfies $\partial f / \partial x = -i \partial f / \partial y$, which, reassuringly, agrees with the first equation in (16.7).

Example 16.4. Consider the elementary function

$$z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

Its real part $u = x^3 - 3xy^2$ and imaginary part $v = 3x^2y - y^3$ satisfy the Cauchy–Riemann equations (16.22), since

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}.$$

Theorem 16.3 implies that $f(z) = z^3$ is complex differentiable. Not surprisingly, its derivative turns out to be

$$f'(z) = 3z^2 = (3x^2 - 3y^2) + i(6xy) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Fortunately, the complex derivative obeys all of the usual rules that you learned in real-variable calculus. For example,

$$\frac{d}{dz} z^n = n z^{n-1}, \quad \frac{d}{dz} e^{cz} = c e^{cz}, \quad \frac{d}{dz} \log z = \frac{1}{z}, \quad (16.24)$$

[†] In addition to his contributions to complex analysis, partial differential equations and number theory, Bernhard Riemann also was the inventor of Riemannian geometry, which turned out to be absolutely essential for Einstein’s theory of general relativity some 70 years later!

and so on. The power n can even be non-integral or, in view of the identity $z^n = e^{n \log z}$, complex, while c is any complex constant. The exponential formulae (16.17) for the complex trigonometric functions implies that they also satisfy the standard rules

$$\frac{d}{dz} \cos z = -\sin z, \quad \frac{d}{dz} \sin z = \cos z. \quad (16.25)$$

The formulae for differentiating sums, products, ratios, inverses, and compositions of complex functions are all identical to their real counterparts. Thus, thankfully, you don't need to learn any new rules for performing complex differentiation!

Remark: There are many examples of seemingly reasonable functions which do *not* have a complex derivative. The simplest is the complex conjugate function

$$f(z) = \bar{z} = x - iy.$$

Its real and imaginary parts do *not* satisfy the Cauchy–Riemann equations, and hence $\bar{z}$ does *not* have a complex derivative. More generally, any function $f(x, y) = h(z, \bar{z})$ that explicitly depends on the complex conjugate variable $\bar{z}$ is *not* complex-differentiable.

Power Series and Analyticity

The most remarkable feature of complex analysis, which distinguishes it from real function theory, is that the existence of one complex derivative automatically implies the existence of infinitely many! All complex functions $f(z)$ are infinitely differentiable and, in fact, analytic where defined. The reason for this surprising and profound fact will, however, not become evident until we learn the basics of complex integration in Section 16.5. In this section, we shall take analyticity as a given, and investigate some of its principal consequences.

Definition 16.5. A complex function $f(z)$ is called *analytic* at a point z_0 if it has a power series expansion

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (16.26)$$

which converges for all z sufficiently close to z_0 .

Typically, the standard ratio or root tests for convergence of (real) series that you learned in ordinary calculus, [9, 167], can be applied to determine where a given (complex) power series converges. We note that if $f(z)$ and $g(z)$ are analytic at a point z_0 , so is their sum $f(z) + g(z)$, product $f(z)g(z)$ and, provided $g(z_0) \neq 0$, ratio $f(z)/g(z)$.

Example 16.6. All of the real power series found in elementary calculus carry over to the complex versions of the functions. For example,

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (16.27)$$

is the Taylor series for the exponential function based at $z_0 = 0$. A simple application of the ratio test proves that the series converges for all z . On the other hand, the power series

$$\frac{1}{z^2 + 1} = 1 - z^2 + z^4 - z^6 + \cdots = \sum_{k=0}^{\infty} (-1)^k z^{2k}, \quad (16.28)$$

converges inside the unit disk, where $|z| < 1$, and diverges outside, where $|z| > 1$. Again, convergence is established through the ratio test. The ratio test is inconclusive when $|z| = 1$, and we shall leave the more delicate question of precisely where on the unit disk this complex series converges to a more advanced treatment, e.g., [4].

In general, there are three possible options for the domain of convergence of a complex power series (16.26):

- (a) The series converges for all z .
- (b) The series converges inside a disk $|z - z_0| < \rho$ of radius $\rho > 0$ centered at z_0 and diverges for all $|z - z_0| > \rho$ outside the disk. The series may converge at some (but not all) of the points on the boundary of the disk where $|z - z_0| = \rho$.
- (c) The series only converges, trivially, at $z = z_0$.

The number ρ is known as the *radius of convergence* of the series. In case (a), we say $\rho = \infty$, while in case (c), $\rho = 0$, and the series does *not* represent an analytic function. An example with $\rho = 0$ is the power series $\sum n! z^n$. In the intermediate case (b), determining precisely where on the boundary of the convergence disk the power series converges is quite delicate, and will not be pursued here. The proof of this result can be found in Exercise ■; see also [4, 95] for further details.

Remarkably, the radius of convergence for the power series of a known analytic function $f(z)$ can be determined by inspection, without recourse to any fancy convergence tests! Namely, ρ is equal to the distance from z_0 to the nearest *singularity* of $f(z)$, meaning a point where the function fails to be analytic. This explains why the Taylor series of e^z converges everywhere, while that of $(z^2 + 1)^{-1}$ only converges inside the unit disk. Indeed e^z is analytic for all z and has no singularities; therefore the radius of convergence of its power series — centered at any point z_0 — is equal to $\rho = \infty$. On the other hand, the function

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z + i)(z - i)}$$

has singularities at $z = \pm i$, and so the series (16.28) has radius of convergence $\rho = 1$, which is the distance from $z_0 = 0$ to the singularities. Thus, the extension of the theory of power series to the complex plane serves to explain the apparent mystery of why, as a real function, $(1 + x^2)^{-1}$ is well-defined and analytic for all real x , but its power series only converges on the interval $(-1, 1)$. It is the *complex* singularities that prevent its convergence when $|x| > 1$. If we expand $(z^2 + 1)^{-1}$ in a power series at some other point, say $z_0 = 1 + 2i$, then we need to determine which singularity is closest. We compute $|i - z_0| = |-1 - i| = \sqrt{2}$, while $|-i - z_0| = |-1 - 3i| = \sqrt{10}$, and so the radius of convergence $\rho = \sqrt{2}$ is the smaller. Thus we can determine the radius of convergence without any explicit formula for its (rather complicated) Taylor expansion at $z_0 = 1 + 2i$.

There are, in fact, only three possible types of singularities of a complex function $f(z)$:

(i) *Pole*. A singular point $z = z_0$ is called a *pole* of *order* $n > 0$ if and only if

$$f(z) = \frac{h(z)}{(z - z_0)^n}, \quad (16.29)$$

where $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$. The simplest example of such a function is $f(z) = a(z - z_0)^{-n}$ for $a \neq 0$ a complex constant.

- (ii) *Branch point*. We have already encountered the two basic types: *algebraic branch points*, such as the function $\sqrt[n]{z}$ at $z_0 = 0$, and *logarithmic branch points* such as $\log z$ at $z_0 = 0$. The *degree* of the branch point is n in the first case and ∞ in the second.
- (iii) *Essential singularity*. By definition, a singularity is *essential* if it is not a pole or a branch point. The simplest example is the essential singularity at $z_0 = 0$ of the function $e^{1/z}$. Details are left as an Exercise ■.

Example 16.7. The complex function

$$f(z) = \frac{e^z}{z^3 - z^2 - 5z - 3} = \frac{e^z}{(z - 3)(z + 1)^2}$$

is analytic everywhere except for singularities at the points $z = 3$ and $z = -1$, where its denominator vanishes. Since

$$f(z) = \frac{h_1(z)}{z - 3}, \quad \text{where} \quad h_1(z) = \frac{e^z}{(z + 1)^2}$$

is analytic at $z = 3$ and $h_1(3) = \frac{1}{16}e^3 \neq 0$, we see that $z = 3$ is a simple (order 1) pole for $f(z)$. Similarly,

$$f(z) = \frac{h_2(z)}{(z + 1)^2}, \quad \text{where} \quad h_2(z) = \frac{e^z}{z - 3}$$

is analytic at $z = -1$ with $h_2(-1) = -\frac{1}{4}e^{-1} \neq 0$, we see that the point $z = -1$ is a double (order 2) pole.

A complicated complex function can have a variety of singularities. For example, the function

$$f(z) = \frac{\sqrt[3]{z + 2} e^{-1/z}}{z^2 + 1} \quad (16.30)$$

has simple poles at $z = \pm i$, a branch point of degree 3 at $z = -2$, and an essential singularity at $z = 0$.

As in the real case, and unlike Fourier series, convergent power series can always be repeatedly term-wise differentiated. Therefore, given the convergent series (16.26), we have the corresponding series

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + 4a_4(z - z_0)^3 + \cdots = \sum_{n=0}^{\infty} (n + 1)a_{n+1}(z - z_0)^n,$$

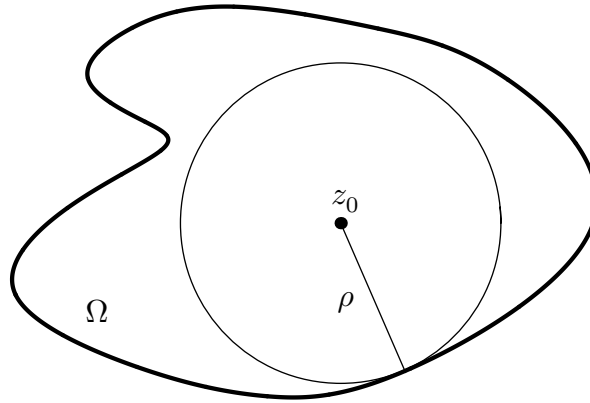


Figure 16.7. Radius of Convergence.

$$\begin{aligned}
 f''(z) &= 2a_2 + 6a_3(z - z_0) + 12a_4(z - z_0)^2 + 20a_5(z - z_0)^3 + \cdots \\
 &= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}(z - z_0)^n, \quad (16.31)
 \end{aligned}$$

and so on, for its derivatives. The proof that the differentiated series have the same radius of convergence can be found in [4, 156]. As a consequence, we deduce the following important result.

Theorem 16.8. *Any analytic function is infinitely differentiable.*

In particular, when we substitute $z = z_0$ into the successively differentiated series, we discover that

$$a_0 = f(z_0), \quad a_1 = f'(z_0), \quad a_2 = \frac{1}{2} f''(z_0),$$

and, in general,

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \quad (16.32)$$

Therefore, a convergent power series (16.26) is, inevitably, the usual *Taylor series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \quad (16.33)$$

for the function $f(z)$ at the point z_0 .

Let us conclude this section by summarizing the fundamental theorem that characterizes complex functions. A complete, rigorous proof relies on complex integration theory, which is the topic of Section 16.5.

Theorem 16.9. *Let $\Omega \subset \mathbb{C}$ be an open set. The following properties are equivalent:*

- (a) *The function $f(z)$ has a continuous complex derivative $f'(z)$ for all $z \in \Omega$.*
- (b) *The real and imaginary parts of $f(z)$ have continuous partial derivatives and satisfy the Cauchy–Riemann equations (16.22) in Ω .*

- (c) The function $f(z)$ is analytic for all $z \in \Omega$, and so is infinitely differentiable and has a convergent power series expansion at each point $z_0 \in \Omega$. The radius of convergence ρ is at least as large as the distance from z_0 to the boundary $\partial\Omega$; see Figure 16.7.

From now on, we reserve the term *complex function* to signify one that satisfies the conditions of Theorem 16.9. Sometimes one of the equivalent adjectives “analytic” or “holomorphic”, is added for emphasis. From now on, all complex functions are assumed to be analytic everywhere on their domain of definition, except, possibly, at certain isolated singularities.

16.3. Harmonic Functions.

We began this section by motivating the analysis of complex functions through applications to the solution of the two-dimensional Laplace equation. Let us now formalize the precise relationship between the two subjects.

Theorem 16.10. *If $f(z) = u(x, y) + i v(x, y)$ is any complex analytic function, then its real and imaginary parts, $u(x, y), v(x, y)$, are both harmonic functions.*

Proof: Differentiating[†] the Cauchy–Riemann equations (16.22), and invoking the equality of mixed partial derivatives, we find that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} \right) = \frac{\partial}{\partial y} \left(-\frac{\partial u}{\partial y} \right) = -\frac{\partial^2 u}{\partial y^2}.$$

Therefore, u is a solution to the Laplace equation $u_{xx} + u_{yy} = 0$. The proof for v is similar. Q.E.D.

Thus, every complex function gives rise to two harmonic functions. It is, of course, of interest to know whether we can invert this procedure. Given a harmonic function $u(x, y)$, does there exist a harmonic function $v(x, y)$ such that $f = u + i v$ is a complex analytic function? If so, the harmonic function $v(x, y)$ is known as a *harmonic conjugate* to u . The harmonic conjugate is found by solving the Cauchy–Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \quad (16.34)$$

which, for a prescribed function $u(x, y)$, constitutes an inhomogeneous linear system of partial differential equations for $v(x, y)$. As such, it is usually not hard to solve, as the following example illustrates.

Example 16.11. As the reader can verify, the harmonic polynomial

$$u(x, y) = x^3 - 3x^2y - 3xy^2 + y^3$$

[†] Theorem 16.9 allows us to differentiate u and v as often as desired.

satisfies the Laplace equation everywhere. To find a harmonic conjugate, we solve the Cauchy–Riemann equations (16.34). First of all,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3x^2 + 6xy - 3y^2,$$

and hence, by direct integration with respect to x ,

$$v(x, y) = x^3 + 3x^2y - 3xy^2 + h(y),$$

where $h(y)$ — the “constant of integration” — is a function of y alone. To determine h we substitute our formula into the second Cauchy–Riemann equation:

$$3x^2 - 6xy + h'(y) = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2.$$

Therefore, $h'(y) = -3y^2$, and so $h(y) = -y^3 + c$, where c is a real constant. We conclude that every harmonic conjugate to $u(x, y)$ has the form

$$v(x, y) = x^3 + 3x^2y - 3xy^2 - y^3 + c.$$

Note that the corresponding complex function

$$\begin{aligned} u(x, y) + i v(x, y) &= (x^3 - 3x^2y - 3xy^2 + y^3) + i(x^3 + 3x^2y - 3xy^2 - y^3 + c) \\ &= (1 - i)z^3 + c \end{aligned}$$

turns out to be a complex cubic polynomial.

Remark: On a connected domain, all harmonic conjugates to a given function $u(x, y)$ only differ by a constant: $\tilde{v}(x, y) = v(x, y) + c$; see Exercise ■.

Although most harmonic functions have harmonic conjugates, unfortunately this is not always the case. Interestingly, the existence or non-existence of a harmonic conjugate can depend on the underlying geometry of the domain of definition of the function. If the domain is simply-connected, and so contains no holes, then one can *always* find a harmonic conjugate. Otherwise, if the domain of definition Ω of our harmonic function $u(x, y)$ is not simply-connected, then there may not exist a single-valued harmonic conjugate $v(x, y)$ to serve as the imaginary part of a complex function $f(z)$.

Example 16.12. The simplest example where the latter possibility occurs is the logarithmic potential

$$u(x, y) = \log r = \frac{1}{2} \log(x^2 + y^2).$$

This function is harmonic on the non-simply-connected domain $\Omega = \mathbb{C} \setminus \{0\}$, but it is not the real part of any single-valued complex function. Indeed, according to (16.18), the logarithmic potential is the real part of the multiply-valued complex logarithm $\log z$, and so its harmonic conjugate[†] is $\text{ph } z = \theta$, which cannot be consistently and continuously

[†] We can, by a previous remark, add in any constant to the harmonic conjugate, but this does not affect the subsequent argument.

defined on all of Ω . On the other hand, restricting z to a simply connected subdomain $\tilde{\Omega} \not\ni 0$ allows us to select a continuous, single-valued branch of the angle $\theta = \text{ph } z$, and so $\log r$ does have a genuine harmonic conjugate on $\tilde{\Omega}$.

The harmonic function

$$u(x, y) = \frac{x}{x^2 + y^2}$$

is also defined on the same non-simply-connected domain $\Omega = \mathbb{C} \setminus \{0\}$ with a singularity at $x = y = 0$. In this case, there is a single valued harmonic conjugate, namely

$$v(x, y) = -\frac{y}{x^2 + y^2},$$

which is defined on all of Ω . Indeed, according to (16.12), these functions define the real and imaginary parts of the complex function $u + i v = 1/z$. Alternatively, one can directly check that they satisfy the Cauchy–Riemann equations (16.22).

Remark: On the “punctured” plane $\Omega = \mathbb{C} \setminus \{0\}$, the logarithmic potential is, in a sense, the only counterexample that prevents a harmonic conjugate from being constructed. It can be shown, [XC], that if $u(x, y)$ is a harmonic function defined on a punctured disk $\Omega_R = \{0 < |z| < R\}$, where $0 < R \leq \infty$, then there exists a constant c such that $\tilde{u}(x, y) = u(x, y) - c \log r$ is also harmonic and possess a single-valued harmonic conjugate $\tilde{v}(x, y)$. As a result, the function $\tilde{f} = \tilde{u} + i\tilde{v}$ is analytic on all of Ω_R , and so our original function $u(x, y)$ is the real part of the multiply-valued analytic function $f(z) = \tilde{f}(z) + c \log z$. We shall use this fact in our later analysis of airfoils.

Theorem 16.13. *Every harmonic function $u(x, y)$ defined on a simply-connected domain Ω is the real part of a complex valued function $f(z) = u(x, y) + i v(x, y)$ which is defined for all $z = x + i y \in \Omega$.*

Proof: We first rewrite the Cauchy–Riemann equations (16.34) in vectorial form as an equation for the gradient of v :

$$\nabla v = \nabla^\perp u, \quad \text{where} \quad \nabla^\perp u = \begin{pmatrix} -u_y \\ u_x \end{pmatrix} \quad (16.35)$$

is the vector field that is everywhere orthogonal to the gradient of u and of the same length:

$$\nabla u \cdot \nabla^\perp u = 0, \quad \|\nabla^\perp u\| = \|\nabla u\|.$$

These properties along with the right hand rule serve to uniquely characterize $\nabla u^\perp$. Thus, the gradient of a harmonic function and that of its harmonic conjugate are mutually orthogonal vector fields having the same Euclidean lengths:

$$\nabla u \cdot \nabla v \equiv 0, \quad \|\nabla u\| \equiv \|\nabla v\|. \quad (16.36)$$

Now, according to Theorem A.8, provided we work on a simply-connected domain, the gradient equation

$$\nabla v = \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

has a solution if and only if the vector field $\mathbf{f}$ satisfies the curl-free constraint

$$\nabla \wedge \mathbf{f} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \equiv 0.$$

In our specific case, the curl of the perpendicular vector field $\nabla u^\perp$ coincides with the divergence of ∇u , which, in turn, coincides with the Laplacian:

$$\nabla \wedge \nabla u^\perp = \nabla \cdot \nabla u = \Delta u = 0, \quad \text{i.e.,} \quad \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(-\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The result is zero because we are assuming that u is harmonic. Equation (A.41) permits us to reconstruct the harmonic conjugate $v(x, y)$ from its gradient ∇v through line integration

$$v(x, y) = \int_C \nabla v \cdot d\mathbf{x} = \int_C \nabla u^\perp \cdot d\mathbf{x} = \int_C \nabla u \cdot \mathbf{n} \, ds, \quad (16.37)$$

where C is any curve connecting a fixed point (x_0, y_0) to (x, y) . Therefore, the harmonic conjugate to a given potential function u can be obtained by evaluating its (path-independent) flux integral (16.37). Q.E.D.

Remark: As a consequence of (16.23) and the Cauchy–Riemann equations (16.34),

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}. \quad (16.38)$$

Thus, the individual components of the gradients ∇u and ∇v appear as the real and imaginary parts of the complex derivative $f'(z)$.

The orthogonality (16.35) of the gradient of a function and of its harmonic conjugate has the following important geometric consequence. Recall, Theorem A.14, that the gradient ∇u of a function $u(x, y)$ points in the normal direction to its *level curves*, that is, the sets $\{u(x, y) = c\}$ where it assumes a fixed constant value. Since ∇v is orthogonal to ∇u , this must mean that ∇v is *tangent* to the level curves of u . Vice versa, ∇v is normal to its level curves, and so ∇u is tangent to the level curves of its harmonic conjugate v . Since their tangent directions ∇u and ∇v are orthogonal, the level curves of the real and imaginary parts of a complex function form a mutually orthogonal system of plane curves — but with one key exception. If we are at a *critical point*, where $\nabla u = \mathbf{0}$, then $\nabla v = \nabla u^\perp = \mathbf{0}$, and the vectors do not define tangent directions. Therefore, the orthogonality of the level curves does not necessarily hold at critical points. It is worth pointing out that, in view of (16.38), the critical points of u are the same as those of v and also the same as the critical points of the corresponding complex function $f(z)$, i.e., those points where its complex derivative vanishes: $f'(z) = 0$.

In Figure 16.8, we illustrate the preceding discussion by plotting the level curves of the real and imaginary parts of the monomials z^2 and z^3 . Note that, except at the origin, where the derivative vanishes, the level curves intersect everywhere at right angles.

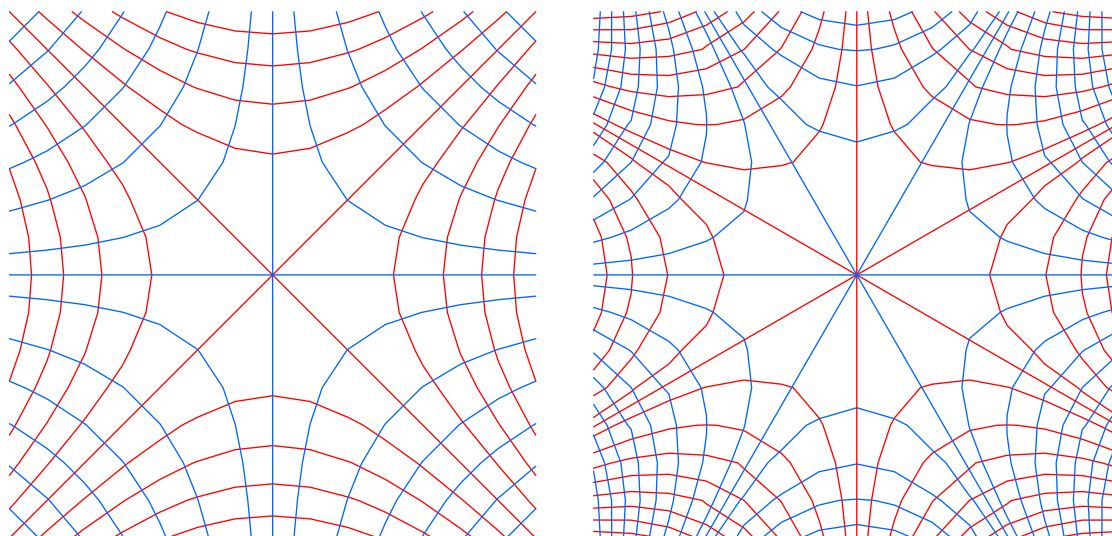


Figure 16.8. Level Curves of the Real and Imaginary Parts of z^2 and z^3 .

Applications to Fluid Mechanics

Consider a planar[†] steady state fluid flow, with velocity vector field

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \quad \text{at the point } \mathbf{x} = (x, y) \in \Omega.$$

Here $\Omega \subset \mathbb{R}^2$ is the domain occupied by the fluid, while the vector $\mathbf{v}(\mathbf{x})$ represents the instantaneous velocity of the fluid at the point $\mathbf{x}$. Recall that the flow is *incompressible* if and only if it has vanishing divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (16.39)$$

Incompressibility means that the fluid volume does not change as it flows. Most liquids, including water, are, for all practical purposes, incompressible. On the other hand, the flow is *irrotational* if and only if it has vanishing curl:

$$\nabla \wedge \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \quad (16.40)$$

Irrotational flows has no vorticity or circulation and model fluids in non-turbulent conditions. In many physical situations, the flow of liquids (and, although less often, gases) is both incompressible and irrotational, which for short, is designated an *ideal fluid flow*.

The two constraints (16.39–40) are almost identical to the Cauchy–Riemann equations (16.22)! The only difference is the change in sign in front of the derivatives of v , but this can be easily remedied by replacing v by its negative $-v$. As a result, we deduce a profound connection between ideal planar fluid flows and complex functions.

[†] See the remarks in Appendix A on the interpretation of a planar fluid flow as the cross-section of a fully three-dimensional fluid motion that does not depend upon the vertical coordinate.

Theorem 16.14. The vector field $\mathbf{v} = (u(x, y), v(x, y))^T$ is the velocity vector of an ideal fluid flow if and only if

$$f(z) = u(x, y) - i v(x, y) \quad (16.41)$$

is a complex analytic function of $z = x + i y$.

Thus, the components $u(x, y)$ and $-v(x, y)$ of the velocity vector field for an ideal fluid are harmonic conjugates. The complex function (16.41) is known as the *complex velocity* of the fluid flow. When applying this result, *do not forget* the minus sign that appears in front of the imaginary part of $f(z)$.

As discussed in Example A.7, the fluid particles will follow the trajectories $z(t) = x(t) + i y(t)$ obtained by integrating the differential equations

$$\frac{dx}{dt} = u(x, y), \quad \frac{dy}{dt} = v(x, y). \quad (16.42)$$

In view of the representation (16.41), we can rewrite the system in complex form

$$\frac{dz}{dt} = \overline{f(z)}. \quad (16.43)$$

In fluid mechanics, the curves[†] parametrized by $z(t)$ are known as the *streamlines*. Each fluid particle's motion $z(t)$ is uniquely prescribed by its position $z(t_0) = z_0 = x_0 + i y_0$ at an initial time t_0 . In particular, if the complex velocity vanishes, $f(z_0) = 0$, then the solution $z(t) \equiv z_0$ to (16.43) is constant, and hence z_0 is a *stagnation point* of the flow. Our steady state assumption, which is reflected in the fact that the ordinary differential equations (16.42) are autonomous, i.e., there is no explicit t dependence, means that, although the fluid is in motion, the stream lines and stagnation point do not change over time. This is a consequence of the standard existence and uniqueness theorems for solutions to ordinary differential equations, to be discussed in detail in Chapter 20.

Example 16.15. The simplest example is when the velocity is constant, corresponding to a uniform, steady flow. Consider first the case

$$f(z) = 1,$$

which corresponds to the horizontal velocity vector field $\mathbf{v} = (1, 0)^T$. The actual fluid flow is found by integrating the system

$$\dot{z} = 1, \quad \text{or} \quad \dot{x} = 1, \quad \dot{y} = 0.$$

Thus, the solution $z(t) = t + z_0$ represents a uniform horizontal fluid motion whose streamlines are straight lines parallel to the real axis; see Figure 16.9.

Consider next a more general constant velocity

$$f(z) = c = a + i b.$$

[†] See below for more details on complex curves.

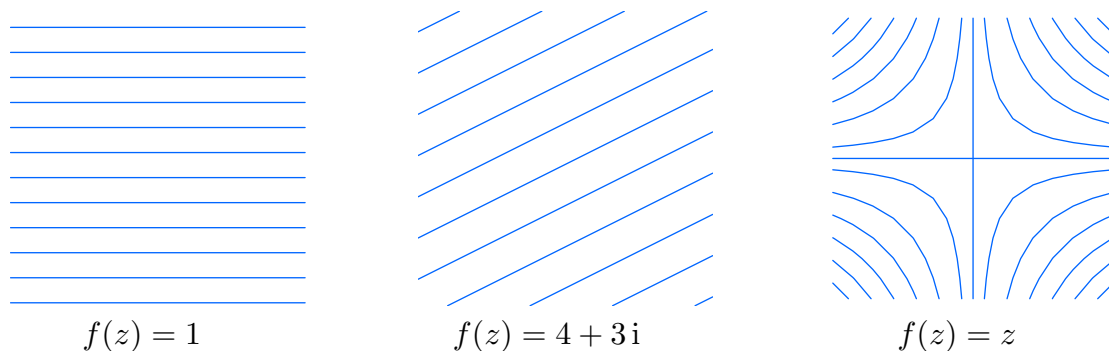


Figure 16.9. Complex Fluid Flows.

The fluid particles will solve the ordinary differential equation

$$\dot{z} = \bar{c} = a - ib, \quad \text{so that} \quad z(t) = \bar{c}t + z_0.$$

The streamlines remain parallel straight lines, but now at an angle $\theta = \text{ph } \bar{c} = -\text{ph } c$ with the horizontal. The fluid particles move along the streamlines at constant speed $|\bar{c}| = |c|$.

The next simplest complex velocity function is

$$f(z) = z = x + iy. \quad (16.44)$$

The corresponding fluid flow is found by integrating the system

$$\dot{z} = \bar{z}, \quad \text{or, in real form,} \quad \dot{x} = x, \quad \dot{y} = -y.$$

The origin $x = y = 0$ is a stagnation point. The trajectories of the nonstationary solutions

$$z(t) = x_0 e^t + iy_0 e^{-t} \quad (16.45)$$

are the hyperbolas $xy = c$, and the positive and negative coordinate semi-axes, as illustrated in Figure 16.9.

On the other hand, if we choose

$$f(z) = -iz = y - ix,$$

then the flow is the solution to

$$\dot{z} = i\bar{z}, \quad \text{or, in real form,} \quad \dot{x} = y, \quad \dot{y} = x.$$

The solutions

$$z(t) = (x_0 \cosh t + y_0 \sinh t) + i(x_0 \sinh t + y_0 \cosh t),$$

move along the hyperbolas (and rays) $x^2 - y^2 = c^2$. Observe that this flow can be obtained by rotating the preceding example by 45° .

In general, a solid object in a fluid flow is characterized by the no-flux condition that the fluid velocity $\mathbf{v}$ is everywhere tangent to the boundary, and hence no fluid flows into or out of the object. As a result, the boundary will consist of streamlines and stagnation points of the idealized fluid flow. For example, the boundary of the upper right quadrant

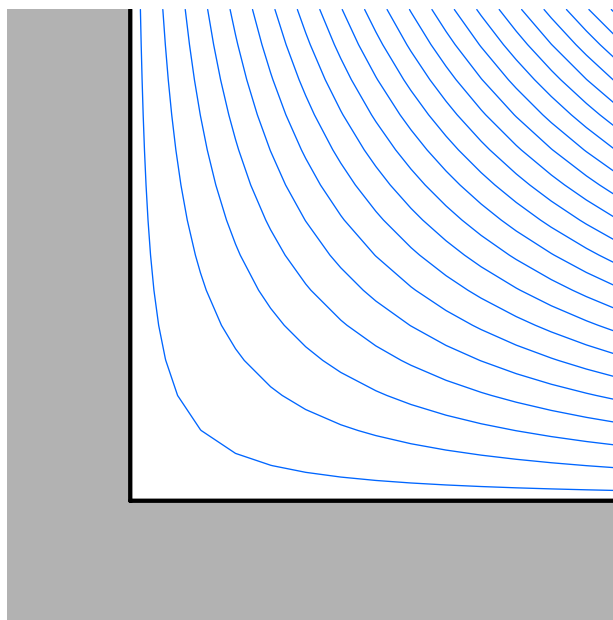


Figure 16.10. Flow Inside a Corner.

$Q = \{x > 0, y > 0\} \subset \mathbb{C}$ consists of the positive x and y axes (along with the origin). Since these are streamlines of the flow with complex velocity (16.44), its restriction to Q represents the flow past a 90° interior corner, which appears in Figure 16.10. The fluid particles move along hyperbolas as they flow past the corner.

Remark: We could also restrict this flow to the domain $\Omega = \mathbb{C} \setminus \{x < 0, y < 0\}$ consisting of three quadrants, corresponding to a 90° exterior corner. However, this flow is not as physically relevant since it has an unrealistic asymptotic behavior at large distances. See Exercise ■ for the “correct” physical flow around an exterior corner.

Now, suppose that the complex velocity $f(z)$ admits a complex anti-derivative, i.e., a complex analytic function

$$\chi(z) = \varphi(x, y) + i\psi(x, y) \quad \text{that satisfies} \quad \frac{d\chi}{dz} = f(z). \quad (16.46)$$

Using the formula (16.23) for the complex derivative,

$$\frac{d\chi}{dz} = \frac{\partial\varphi}{\partial x} - i \frac{\partial\varphi}{\partial y} = u - iv, \quad \text{so} \quad \frac{\partial\varphi}{\partial x} = u, \quad \frac{\partial\varphi}{\partial y} = v.$$

Thus, $\nabla\varphi = \mathbf{v}$, and hence the real part $\varphi(x, y)$ of the complex function $\chi(z)$ defines a *velocity potential* for the fluid flow. For this reason, the anti-derivative $\chi(z)$ is known as a *complex potential function* for the given fluid velocity field.

Since the complex potential is analytic, its real part — the potential function — is harmonic, and therefore satisfies the Laplace equation $\Delta\varphi = 0$. Conversely, any harmonic function can be viewed as the potential function for some fluid flow. The real fluid velocity is its gradient $\mathbf{v} = \nabla\varphi$. The harmonic conjugate $\psi(x, y)$ to the velocity potential also plays an important role, and, in fluid mechanics, is known as the *stream function*. It also satisfies

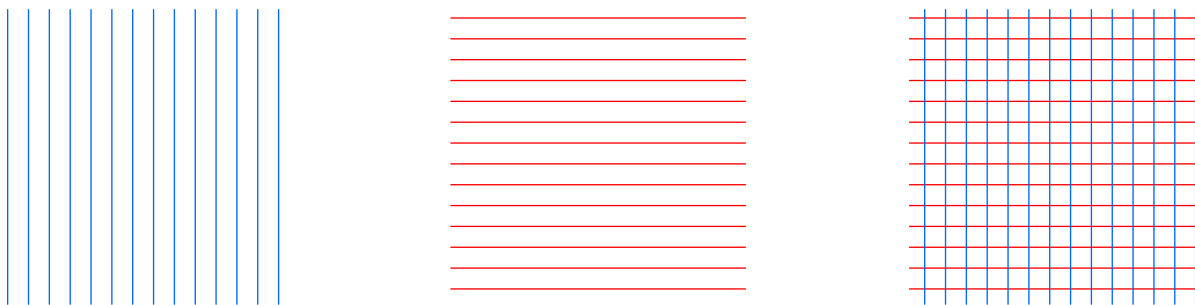


Figure 16.11. Equipotentials and Streamlines for $\chi(z) = z$.

the Laplace equation $\Delta\psi = 0$, and the potential and stream function are related by the Cauchy–Riemann equations (16.22). Thus, the potential and stream function satisfy

$$\frac{\partial\varphi}{\partial x} = u = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y} = v = -\frac{\partial\psi}{\partial x}. \quad (16.47)$$

The level sets of the velocity potential, $\{\varphi(x, y) = c\}$ where $c \in \mathbb{R}$ is fixed, are known as *equipotential curves*. The velocity vector $\mathbf{v} = \nabla\varphi$ points in the normal direction to the equipotentials. On the other hand, as we noted above, $\mathbf{v} = \nabla\varphi$ is tangent to the level curves $\{\psi(x, y) = d\}$ of its harmonic conjugate stream function. But $\mathbf{v}$ is the velocity field, and so tangent to the streamlines followed by the fluid particles. Thus, these two systems of curves must coincide, and we infer that *the level curves of the stream function are the streamlines of the flow*, whence its name! Summarizing, for an ideal fluid flow, the equipotentials $\{\varphi = c\}$ and streamlines $\{\psi = d\}$ form mutually orthogonal systems of plane curves. The fluid velocity $\mathbf{v} = \nabla\varphi$ is tangent to the stream lines and normal to the equipotentials, whereas the gradient of the stream function $\nabla\psi$ is tangent to the equipotentials and normal to the streamlines.

The discussion in the preceding paragraph implicitly relied on the fact that the velocity is nonzero, $\mathbf{v} = \nabla\varphi \neq 0$, which means we are not at a *stagnation point*, where the fluid is not moving. While streamlines and equipotentials might begin or end at a stagnation point, there is no guarantee, and, indeed, in general it is not the case that they meet at mutually orthogonal directions there.

Example 16.16. The simplest example of a complex potential function is

$$\chi(z) = z = x + iy.$$

Thus, the velocity potential is $\varphi(x, y) = x$, while its harmonic conjugate stream function is $\psi(x, y) = y$. The complex derivative of the potential is the complex velocity,

$$f(z) = \frac{d\chi}{dz} = 1,$$

which corresponds to the uniform horizontal fluid motion considered first in Example 16.15. Note that the horizontal stream lines coincide with the level sets $\{y = d\}$ of the stream function, whereas the equipotentials $\{x = c\}$ are the orthogonal system of vertical lines; see Figure 16.11.

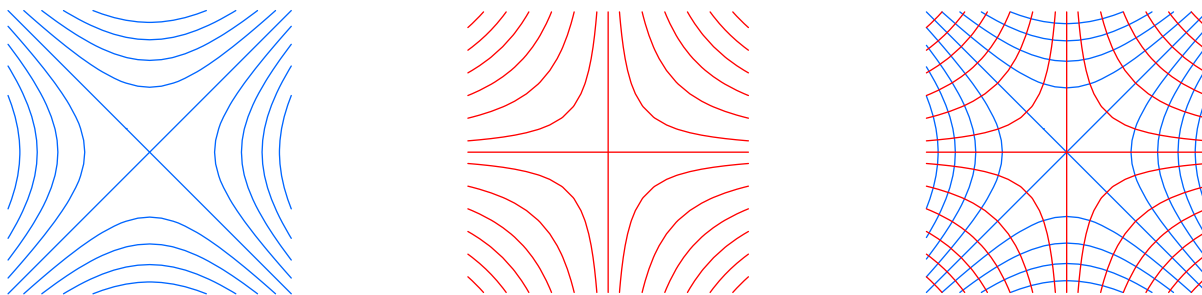


Figure 16.12. Equipotentials and Streamlines for $\chi(z) = \frac{1}{2} z^2$.

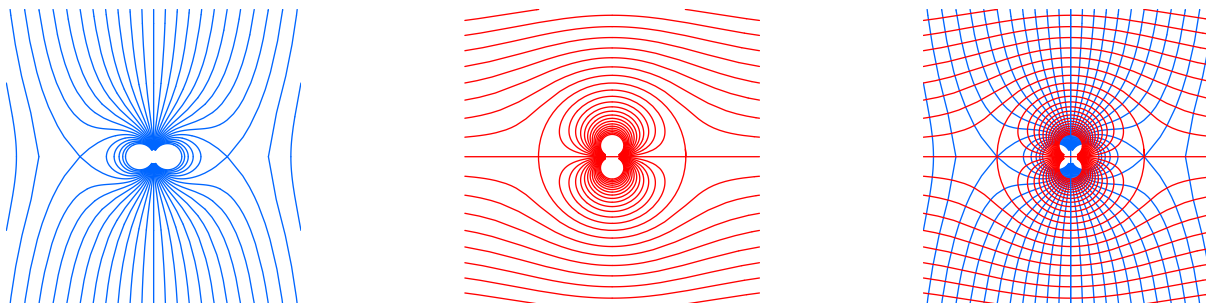


Figure 16.13. Equipotentials and Streamlines for $z + \frac{1}{z}$.

Next, consider the complex potential function

$$\chi(z) = \frac{1}{2} z^2 = \frac{1}{2} (x^2 - y^2) + i x y.$$

The associated complex velocity

$$f(z) = \chi'(z) = z = x + i y$$

leads to the hyperbolic flow (16.45). The hyperbolic streamlines $x y = d$ are the level curves of the stream function $\psi(x, y) = x y$. The equipotential lines $\frac{1}{2} (x^2 - y^2) = c$ form a system of orthogonal hyperbolas. Figure 16.12 shows (some of) the equipotentials in the first plot, the stream lines in the second, and combines them together in the third picture.

Example 16.17. *Flow Around a Disk.* Consider the complex potential function

$$\chi(z) = z + \frac{1}{z} = \left(x + \frac{x}{x^2 + y^2} \right) + i \left(y - \frac{y}{x^2 + y^2} \right). \quad (16.48)$$

The corresponding complex fluid velocity is

$$f(z) = \frac{d\chi}{dz} = 1 - \frac{1}{z^2} = 1 - \frac{x^2 - y^2}{(x^2 + y^2)^2} + i \frac{2xy}{(x^2 + y^2)^2}. \quad (16.49)$$

The equipotential curves and streamlines are plotted in Figure 16.13. The points $z = \pm 1$ are stagnation points of the flow, while $z = 0$ is a singularity. In particular, fluid particles

that move along the positive x axis approach the leading stagnation point $z = 1$ as $t \rightarrow \infty$. Note that the streamlines

$$\psi(x, y) = y - \frac{y}{x^2 + y^2} = d$$

are asymptotically horizontal at large distances, and hence, far away from the origin, the flow is indistinguishable from uniform horizontal motion with complex velocity $f(z) \equiv 1$.

The level curve for the particular value $d = 0$ consists of the unit circle $|z| = 1$ and the real axis $y = 0$. In particular, the unit circle $|z| = 1$ consists of semicircular two stream lines and the two stagnation points. The flow velocity vector field $\mathbf{v} = \nabla\varphi$ is everywhere tangent to the unit circle, and hence satisfies the no flux condition along the boundary of the unit disk. Thus, we can interpret (16.49), when restricted to the domain $\Omega = \{|z| > 1\}$, as the complex velocity of a uniformly moving fluid around the outside of a solid circular disk of radius 1. In three dimensions, this would correspond to the steady flow of a fluid around a solid cylinder; see Figure fcy1■.

In this section, we have focused on the fluid mechanical roles of a harmonic function and its conjugate. An analogous interpretation applies when $\varphi(x, y)$ represents an electromagnetic potential function; the level curves of its harmonic conjugate $\psi(x, y)$ are the paths followed by charged particles under the electromotive force field $\mathbf{v} = \nabla\varphi$. Similarly, if $\varphi(x, y)$ represents the equilibrium temperature distribution in a planar domain, its level lines represent the isotherms or curves of constant temperature, while the level lines of its harmonic conjugate are the curves of heat flow, whose mutual orthogonality was already noted in Appendix A. Finally, if $\varphi(x, y)$ represents the height of a deformed membrane, then its level curves are the contour lines of elevation. The level curves of its harmonic conjugate are the curves of steepest descent along the membrane, i.e., the routes followed by, say, water flowing down the membrane.

16.4. Conformal Mapping.

As we now know, complex functions provide an almost inexhaustible supply of harmonic functions, i.e., solutions to the Laplace equation. Thus, to solve a boundary value problem for Laplace's equation we “merely” need to find the complex function whose real part matches the prescribed boundary conditions. Unfortunately, even for relatively simple domains, this remains a daunting task.

The one case where we do have an explicit solution is that of a circular disk, where the Poisson integral formula (15.48) provides a complete solution to the Dirichlet boundary value problem. (See Exercise ■ for the Neumann problem.) Thus, one evident strategy for solving the corresponding boundary value problem on a more complicated domain is to convert it into the solved case by an inspired change of variables.

The intimate connections between complex analysis and solutions to the Laplace equation inspires us to look at changes of variables defined by complex functions. Thus, we will now interpret a complex analytic function

$$\zeta = g(z) \quad \text{or} \quad \xi + i\eta = p(x, y) + iq(x, y) \quad (16.50)$$

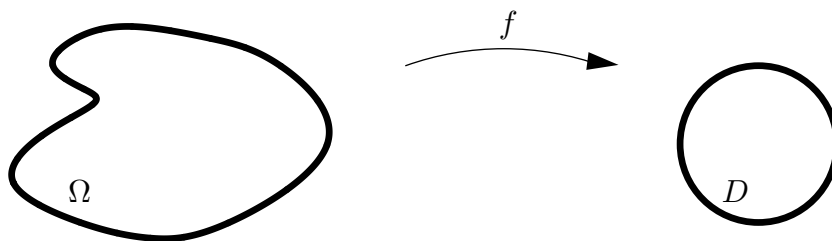


Figure 16.14. Mapping to the Unit Disk.

as a *mapping* that takes a point $z = x + iy$ belonging to a prescribed domain $\Omega \subset \mathbb{C}$ to a point $\zeta = \xi + i\eta$ belonging to the image domain $D = g(\Omega) \subset \mathbb{C}$, as in Figure 16.14. In many cases, D is the unit disk, but may be something else in more general examples. In order unambiguously relate functions on Ω to functions on D , we require that the analytic mapping (16.50) be one-to-one so that each point $\zeta \in D$ comes from a unique point $z \in \Omega$. As a result, the inverse function $z = g^{-1}(\zeta)$ is a well-defined map from D back to Ω , which we assume is also analytic on all of D . The calculus formula for the derivative of the inverse function

$$\frac{d}{d\zeta} g^{-1}(\zeta) = \frac{1}{g'(z)} \quad \text{at} \quad \zeta = g(z), \quad (16.51)$$

which remains valid for complex functions, implies that the derivative of $g(z)$ must be nonzero everywhere in order that $g^{-1}(\zeta)$ be differentiable. This condition,

$$g'(z) \neq 0 \quad \text{at every point} \quad z \in \Omega, \quad (16.52)$$

will play a crucial role in the development of the method. Finally, in order to match the boundary conditions, we will assume that the mapping extends continuously to the boundary $\partial\Omega$ and maps it to the boundary ∂D of the image domain.

Before trying to apply this idea to solve boundary value problems for the Laplace equation, we introduce some of the most important examples of analytic mappings.

Example 16.18. The simplest nontrivial analytic maps are the *translations*

$$\zeta = z + \beta = (x + a) + i(y + b), \quad (16.53)$$

where $\beta = a + ib$ is a fixed complex number. The effect of (16.53) is to translate the entire complex plane in the direction given by the vector $(a, b)^T$. In particular, the translation maps the disk $\Omega = \{|z + c| < 1\}$ of radius 1 and center at $-\beta$ to the unit disk $D = \{|\zeta| < 1\}$.

There are two types of linear analytic transformations. First are the scaling maps

$$\zeta = \rho z = \rho x + i\rho y, \quad (16.54)$$

where $\rho \neq 0$ is a fixed nonzero real number. This maps the disk $|z| < 1/|\rho|$ to the unit disk $|\zeta| < 1$. Second are the rotations

$$\zeta = e^{i\varphi} z = (x \cos \varphi - y \sin \varphi) + i(x \sin \varphi + y \cos \varphi) \quad (16.55)$$

around the origin by a fixed (real) angle φ . This maps the unit disk to itself.

Any non-constant *affine transformation*

$$\zeta = \alpha z + \beta, \quad \alpha \neq 0, \quad (16.56)$$

defines an invertible analytic map on all of $\mathbb{C}$, whose inverse $z = \alpha^{-1}(\zeta - \beta)$ is also affine. Writing $\alpha = \rho e^{i\varphi}$ in polar coordinates, we see that the affine map (16.56) can be viewed as the composition of a rotation (16.55), followed by a scaling (16.54), followed by a translation (16.53). As such, it takes the disk $|\alpha z + \beta| < 1$ of radius $1/|\alpha| = 1/|\rho|$ and center $-\beta/\alpha$ to the unit disk $|\zeta| < 1$.

Example 16.19. A more interesting example is the complex function

$$\zeta = g(z) = \frac{1}{z}, \quad \text{or} \quad \xi = \frac{x}{x^2 + y^2}, \quad \eta = -\frac{y}{x^2 + y^2}, \quad (16.57)$$

which defines an *inversion*[†] of the complex plane. The inversion is a one-to-one analytic map everywhere except at the origin $z = 0$; indeed $g(z)$ is its own inverse: $g^{-1}(\zeta) = 1/\zeta$. Note that $g'(z) = -1/z^2$ is never zero, and so the derivative condition (16.52) is satisfied everywhere. Note that $|\zeta| = 1/|z|$, while $\text{ph } \zeta = -\text{ph } z$. Thus, if $\Omega = \{|z| > \rho\}$ denotes the exterior of the circle of radius ρ , then the image points $\zeta = 1/z$ satisfy $|\zeta| = 1/|z|$, and hence the image domain is the *punctured disk* $D = \{0 < |\zeta| < 1/\rho\}$. In particular, the inversion maps the outside of the unit disk to its inside, but with the origin removed, and vice versa. The reader may enjoy seeing what the inversion does to other domains, e.g., the unit square $0 < x, y < 1$.

Example 16.20. The complex exponential

$$\zeta = g(z) = e^z, \quad \text{or} \quad \xi = e^x \cos y, \quad \eta = e^x \sin y, \quad (16.58)$$

satisfies the condition

$$g'(z) = e^z \neq 0$$

everywhere. Nevertheless, it is *not* one-to-one because $e^{z+2\pi i} = e^z$, and so all points differing by an integer multiple of $2\pi i$ are mapped to the same point.

Under the exponential map, the horizontal line $\text{Im } z = b$ is mapped to the curve $\zeta = e^{x+ib} = e^x(\cos b + i \sin b)$, which, as x varies from $-\infty$ to ∞ , traces out the ray emanating from the origin that makes an angle $\text{ph } \zeta = b$ with the real axis. Therefore, the exponential map will map a horizontal strip

$$S_{a,b} = \{a < \text{Im } z < b\}$$

to a wedge-shaped domain

$$\Omega_{a,b} = \{a < \text{ph } \zeta < b\},$$

and is one-to-one provided $|b - a| < 2\pi$. In particular, the horizontal strip

$$S_{-\pi/2, \pi/2} = \{-\tfrac{1}{2}\pi < \text{Im } z < \tfrac{1}{2}\pi\}$$

[†] This is slightly different than the real inversion (15.75); see Exercise ■.

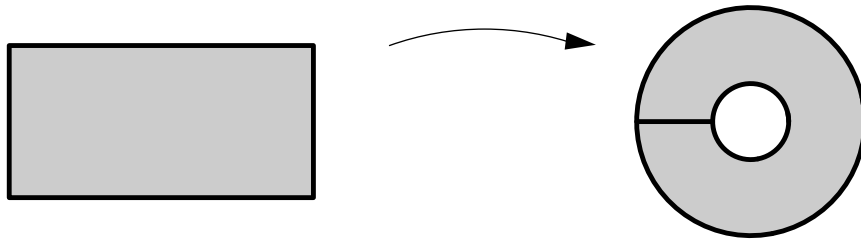


Figure 16.15. The mapping $\zeta = e^z$.

of width π centered around the real axis is mapped, in a one-to-one manner, to the right half plane

$$R = \Omega_{-\pi/2, \pi/2} = \left\{ -\frac{1}{2}\pi < \text{ph } \zeta < \frac{1}{2}\pi \right\} = \left\{ \text{Im } \zeta > 0 \right\},$$

while the horizontal strip $S_{-\pi, \pi} = \left\{ -\pi < \text{Im } z < \pi \right\}$ of width 2π is mapped onto the domain

$$\Omega_* = \Omega_{-\pi, \pi} = \left\{ -\pi < \text{ph } \zeta < \pi \right\} = \mathbb{C} \setminus \left\{ \text{Im } z = 0, \text{Re } z \leq 0 \right\}$$

obtained by cutting the complex plane along the negative real axis.

On the other hand, vertical lines $\text{Re } z = a$ are mapped to circles $|\zeta| = e^a$. Thus, a vertical strip $a < \text{Re } z < b$ is mapped to an annulus $e^a < |\zeta| < e^b$, albeit many-to-one, since the strip is effectively wrapped around and around the annulus. The rectangle $R = \left\{ a < x < b, -\pi < y < \pi \right\}$ of height 2π is mapped in a one-to-one fashion on an annulus that has been cut along the negative real axis. See Figure 16.15.

Example 16.21. The squaring map

$$\zeta = g(z) = z^2, \quad \text{or} \quad \xi = x^2 - y^2, \quad \eta = 2xy, \quad (16.59)$$

is analytic on all of $\mathbb{C}$, but is not one-to-one. Its inverse is the square root function $z = \sqrt{\zeta}$, which, as we noted in Section 16.1, is doubly-valued, except at the origin $z = 0$. Furthermore, the derivative $g'(z) = 2z$ vanishes at $z = 0$, violating the invertibility condition (16.52). However, once we restrict to a simply connected subdomain Ω that does not contain 0, the function $g(z) = z^2$ does define a one-to-one mapping, whose inverse $z = g^{-1}(\zeta) = \sqrt{\zeta}$ is a well-defined, analytic and single-valued branch of the square root function.

The effect of the squaring map on a point z is to square its modulus, $|\zeta| = |z|^2$, while doubling its angle, $\text{ph } \zeta = \text{ph } z^2 = 2 \text{ph } z$. Thus, for example, the upper right quadrant

$$Q = \left\{ x > 0, y > 0 \right\} = \left\{ 0 < \text{ph } z < \frac{1}{2}\pi \right\}$$

is mapped onto the upper half plane

$$U = g(Q) = \left\{ \eta = \text{Im } \zeta > 0 \right\} = \left\{ 0 < \text{ph } \zeta < \pi \right\}.$$

The inverse function maps a point $\zeta \in U$ back to its unique square root $z = \sqrt{\zeta}$ that lies in the quadrant Q . Similarly, a quarter disk

$$Q_\rho = \left\{ 0 < |z| < \rho, 0 < \text{ph } z < \frac{1}{2}\pi \right\}$$

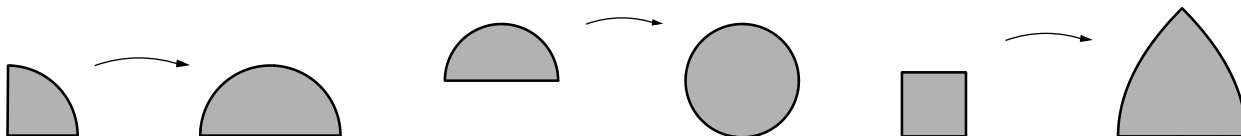


Figure 16.16. The Effect of $\zeta = z^2$ on Various Domains.

of radius ρ is mapped to a half disk

$$U_{\rho^2} = g(\Omega) = \{ 0 < |\zeta| < \rho^2, \operatorname{Im} \zeta > 0 \}$$

of radius ρ^2 . On the other hand, the unit square $S = \{ 0 < x < 1, 0 < y < 1 \}$ is mapped to a curvilinear triangular domain, as indicated in Figure 16.16; the edges of the square on the real and imaginary axes map to the two halves of the straight base of the triangle, while the other two edges become its curved sides.

Example 16.22. A particularly important example is the analytic map

$$\zeta = \frac{z-1}{z+1} = \frac{x^2 + y^2 - 1}{(x+1)^2 + y^2} + i \frac{2y}{(x+1)^2 + y^2}, \quad (16.60)$$

where we derived the formulae for its real and imaginary parts in (16.14). The map is one-to-one with analytic inverse

$$z = \frac{1+\zeta}{1-\zeta} = \frac{1-\xi^2-\eta^2}{(1-\xi)^2+\eta^2} + i \frac{2\eta}{(1-\xi)^2+\eta^2}, \quad (16.61)$$

provided $z \neq -1$ and $\zeta \neq 1$. This particular analytic map has the important property of mapping the right half plane $R = \{ x = \operatorname{Re} z > 0 \}$ to the unit disk $D = \{ |\zeta|^2 < 1 \}$. Indeed, by (16.61)

$$|\zeta|^2 = \xi^2 + \eta^2 < 1 \quad \text{if and only if} \quad x = \frac{1-\xi^2-\eta^2}{(1-\xi)^2+\eta^2} > 0.$$

Note that the denominator does not vanish on the interior of the disk.

The complex functions (16.56, 57, 60) are particular examples of *linear fractional transformations*

$$\zeta = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad (16.62)$$

which form one of the most important classes of analytic maps. Here $\alpha, \beta, \gamma, \delta$ are arbitrary complex constants, subject to the restriction

$$\alpha\delta - \beta\gamma \neq 0,$$

since otherwise (16.62) reduces to a trivial constant (and non-invertible) map. (Why?)

Example 16.23. The linear fractional transformation

$$\zeta = \frac{z - \alpha}{\bar{\alpha}z - 1} \quad \text{where} \quad |\alpha| < 1, \quad (16.63)$$

maps the unit disk to itself, moving the origin $z = 0$ to the point $\zeta = \alpha$. To prove this, we note that

$$\begin{aligned} |z - \alpha|^2 &= (z - \alpha)(\bar{z} - \bar{\alpha}) = |z|^2 - \alpha\bar{z} - \bar{\alpha}z + |\alpha|^2, \\ |\bar{\alpha}z - 1|^2 &= (\bar{\alpha}z - 1)(\alpha\bar{z} - 1) = |\alpha|^2|z|^2 - \alpha\bar{z} - \bar{\alpha}z + 1. \end{aligned}$$

Subtracting these two formulae,

$$|z - \alpha|^2 - |\bar{\alpha}z - 1|^2 = (1 - |\alpha|^2)(|z|^2 - 1) < 0, \quad \text{whenever} \quad |z| < 1, \quad |\alpha| < 1.$$

Thus, $|z - \alpha| < |\bar{\alpha}z - 1|$, which implies that

$$|\zeta| = \frac{|z - \alpha|}{|\bar{\alpha}z - 1|} < 1 \quad \text{provided} \quad |z| < 1, \quad |\alpha| < 1,$$

and hence ζ lies within the unit disk.

The rotations (16.55) also map the unit disk to itself, while preserving the origin. It can be proved, [4], that the only invertible analytic mappings that take the unit disk to itself are obtained by composing such a linear fractional transformation with a rotation.

Proposition 16.24. *If $\zeta = g(z)$ is a one-to-one analytic map that takes the unit disk to itself, then*

$$g(z) = e^{i\varphi} \frac{z - \alpha}{\bar{\alpha}z - 1} \quad \text{for some} \quad |\alpha| < 1, \quad 0 \leq \varphi < 2\pi. \quad (16.64)$$

Additional specific properties of linear fractional transformations are outlined in the exercises.

Conformality

A remarkable geometrical characterization of complex analytic functions is that, at non-critical points, they preserve angles. The mathematical term for this property is *conformal mapping*. Conformality makes sense for any inner product space, although in practice one usually deals with Euclidean space equipped with the standard dot product.

Definition 16.25. A function $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called *conformal* if it preserves angles.

But what does it mean to “preserve angles”? In the Euclidean norm, the angle between two vectors is defined by their dot product, as in (3.20). However, most analytic maps are nonlinear, and so will not map vectors to vectors since they will typically map straight lines to curves. However, if we interpret “angle” to mean the angle between two curves, as illustrated in Figure 16.17, then we can make sense of the conformality requirement. Consequently, in order to realize complex functions as conformal maps, we first need to understand their effect on curves.

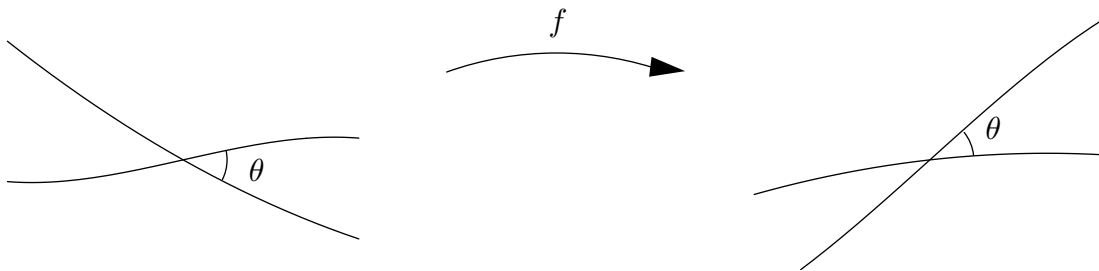


Figure 16.17. A Conformal Map.

In general, a *curve* $C \in \mathbb{C}$ in the complex plane is parametrized by a complex-valued function

$$z(t) = x(t) + i y(t), \quad a < t < b, \quad (16.65)$$

that depends on a real parameter t . Note that there is no essential difference between a complex plane curve (16.65) and a real plane curve, as in (A.1); we have merely switched from vector notation $\mathbf{x}(t) = (x(t), y(t))^T$ to complex notation $z(t) = x(t) + i y(t)$. All the usual vectorial curve terminology (closed, simple, piecewise smooth, etc.), as summarized in Appendix A, is used without any modification here. In particular, the *tangent vector* to the curve can be identified as the complex number $\dot{z}(t) = \dot{x}(t) + i \dot{y}(t)$. Smoothness of the curve is guaranteed by the requirement that $\dot{z}(t) \neq 0$.

Example 16.26. (a) The curve

$$z(t) = e^{it} = \cos t + i \sin t, \quad \text{for } 0 \leq t \leq 2\pi,$$

parametrizes the unit circle $|z| = 1$ in the complex plane, which is a simple closed curve. Its complex tangent $\dot{z}(t) = i e^{it} = i z(t)$ is obtained by rotating z through 90° .

(b) The complex curve

$$z(t) = \cosh t + i \sinh t = \frac{1+i}{2} e^t + \frac{1-i}{2} e^{-t}, \quad -\infty < t < \infty,$$

parametrizes the right hand branch of the hyperbola

$$\operatorname{Re} z^2 = x^2 - y^2 = 1.$$

The complex tangent vector is $\dot{z}(t) = \sinh t + i \cosh t = i \bar{z}(t)$.

In order to better understand curve geometry, it will help to rewrite the tangent $\dot{z}$ in polar coordinates. We interpret the curve as the motion of a particle in the complex plane, so that $z(t)$ is the position of the particle at time t , and the tangent $\dot{z}(t)$ its instantaneous velocity. The modulus of the tangent, $|\dot{z}| = \sqrt{\dot{x}^2 + \dot{y}^2}$, indicates the particle's speed, while its phase $\operatorname{ph} \dot{z}$ measures the direction of motion, as measured by the angle that the curve makes with the horizontal; see Figure 16.18.

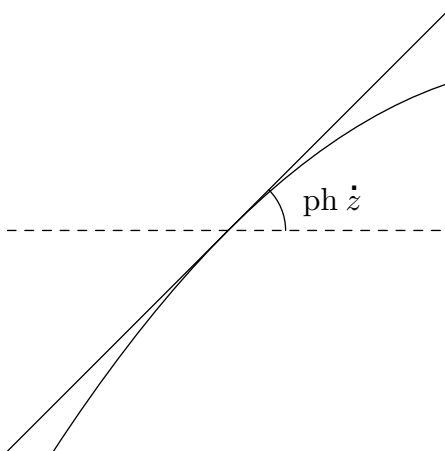


Figure 16.18. Complex Curve and Tangent.

The (signed) angle[†] between two curves is defined as the angle between their tangents at the point of intersection. If the curve C_1 makes an angle $\theta_1 = \text{ph } \dot{z}_1(t_1)$ while the curve C_2 has angle $\theta_2 = \text{ph } \dot{z}_2(t_2)$ at the common point $z = z_1(t_1) = z_2(t_2)$, then the angle θ between C_1 and C_2 at z is their difference

$$\theta = \theta_2 - \theta_1 = \text{ph } \dot{z}_2 - \text{ph } \dot{z}_1 = \text{ph } \left(\frac{\dot{z}_2}{\dot{z}_1} \right). \quad (16.66)$$

Now, suppose we are given an analytic map $\zeta = g(z)$. A curve C parametrized by $z(t)$ will be mapped to a new curve $\Gamma = g(C)$ parametrized by the composition $\zeta(t) = g(z(t))$. The tangent to the image curve is related to that of the original curve by the chain rule:

$$\frac{d\zeta}{dt} = \frac{dg}{dz} \frac{dz}{dt}, \quad \text{or} \quad \dot{\zeta}(t) = g'(z(t)) \dot{z}(t). \quad (16.67)$$

Therefore, the effect of the analytic map on the tangent vector $\dot{z}$ is to multiply it by the complex number $g'(z)$. If the analytic map satisfies our key assumption $g'(z) \neq 0$, then $\dot{\zeta} \neq 0$, and so the image curve is guaranteed to be smooth.

According to equation (16.67),

$$|\dot{\zeta}| = |g'(z) \dot{z}| = |g'(z)| |\dot{z}|. \quad (16.68)$$

Thus, the speed of motion along the new curve $\zeta(t)$ is multiplied by a factor $\rho = |g'(z)| > 0$. The magnification factor ρ depends only upon the point z and not how the curve passes through it. All curves passing through the point z are speeded up (or slowed down if $\rho < 1$) by the same factor! Similarly, the angle that the new curve makes with the horizontal is given by

$$\text{ph } \dot{\zeta} = \text{ph}(g'(z) \dot{z}) = \text{ph } g'(z) + \text{ph } \dot{z}, \quad (16.69)$$

[†] This means that the angle is defined up to a multiple of 2π , and so we distinguish between positive and negative angles between curves. The angle defined by an inner product is necessarily positive. In fact, signed angles only make sense in two dimensions, since a three-dimensional rotation of the curves can change positive to negative angles; see Exercise ■ for further details.

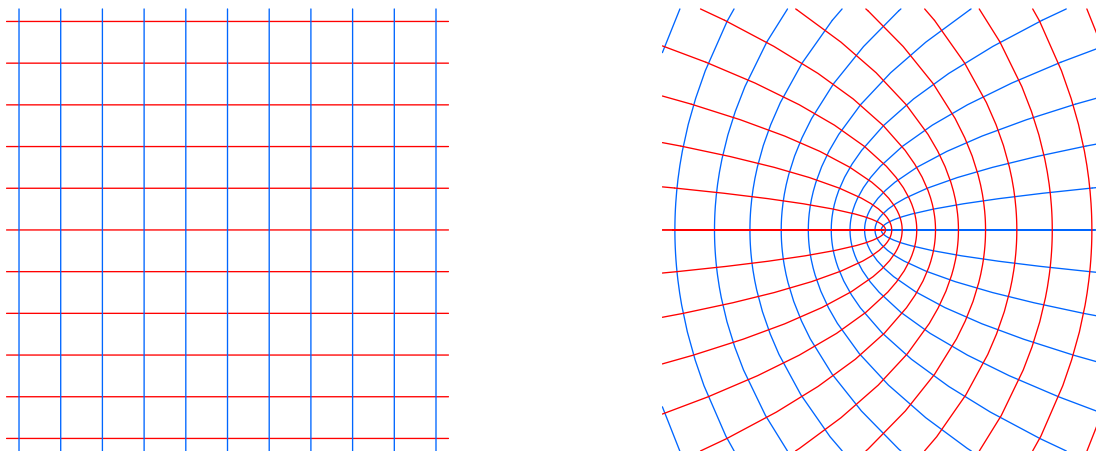


Figure 16.19. Conformality of z^2 .

since the phase of the product of two complex numbers is the sum of their individual phases, (3.82). Therefore, the tangent angle of the curve is increased by an amount $\phi = \text{ph } g'(z)$, which means that the tangent is been rotated through an angle ϕ . Again, the increase in tangent angle only depends on the point z , and all curves passing through z are rotated by the same amount ϕ . As a result, the angle between any two curves is preserved. More precisely, if C_1 is at angle θ_1 and C_2 at angle θ_2 at a point of intersection, then their images $\Gamma_1 = g(C_1)$ and $\Gamma_2 = g(C_2)$ are at angles $\psi_1 = \theta_1 + \phi$ and $\psi_2 = \theta_2 + \phi$. The angle between the two image curves is the difference

$$\psi_2 - \psi_1 = (\theta_2 + \phi) - (\theta_1 + \phi) = \theta_2 - \theta_1,$$

which is *the same* as the angle between the original curves. This establishes the conformality or angle-preservation property of analytic maps.

Theorem 16.27. *If $\zeta = g(z)$ is an analytic function and $g'(z) \neq 0$, then g defines a conformal map.*

Remark: The converse is also valid: Every planar conformal map comes from a complex analytic function with nonvanishing derivative. A proof is outlined in Exercise ■.

The conformality of analytic functions is all the more surprising when one revisits elementary examples. In Example 16.21, we discovered that the function $w = z^2$ maps a quarter plane to a half plane, and therefore *doubles* the angle between the coordinate axes at the origin! Thus $g(z) = z^2$ is most definitely not conformal at $z = 0$. The explanation is, of course, that $z = 0$ is a critical point, $g'(0) = 0$, and Theorem 16.27 only guarantees conformality when the derivative is nonzero. Amazingly, the map preserves angles everywhere else! Somehow, the angle at the origin is doubled, while angles at all nearby points are preserved. Figure 16.19 illustrates this remarkable and counter-intuitive feat. The left hand figure shows the coordinate grid, while on the right are the images of the horizontal and vertical lines under the map z^2 . Note that, except at the origin, the image curves continue to meet at 90° angles, in accordance with conformality.

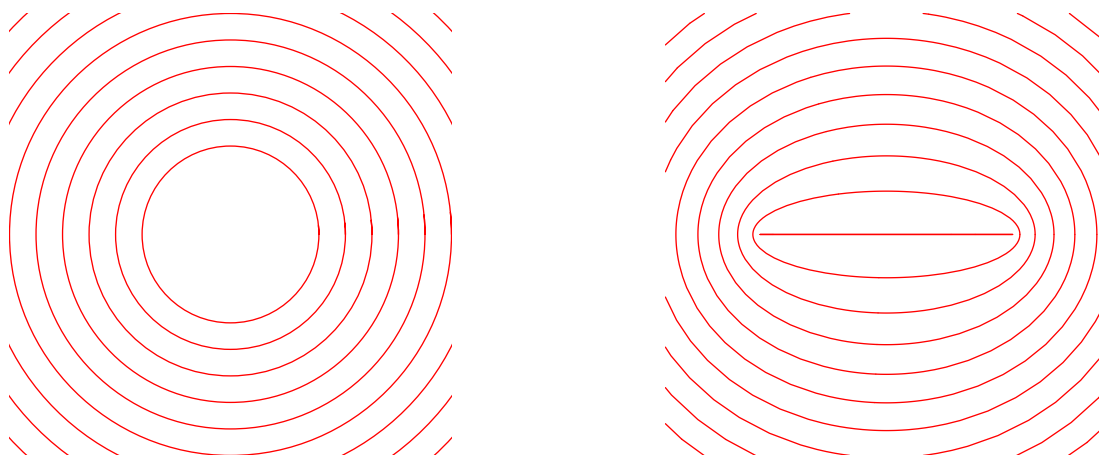


Figure 16.20. The Joukowski Map.

Example 16.28. A particularly interesting conformal transformation is given by the *Joukowski map*

$$\zeta = \frac{1}{2} \left(z + \frac{1}{z} \right). \quad (16.70)$$

It is used in the study of flows around airplane wings, and named after the pioneering Russian aero- and hydro-dynamics researcher Nikolai Zhukovskii (Joukowski). Since

$$\frac{d\zeta}{dz} = \frac{1}{2} \left(1 - \frac{1}{z^2} \right) = 0 \quad \text{if and only if} \quad z = \pm 1,$$

the Joukowski map is conformal except at the critical points $z = \pm 1$, as well as at the singularity $z = 0$ where it is not defined.

If $z = e^{i\theta}$ lies on the unit circle, then

$$\zeta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) = \cos \theta,$$

lies on the real axis, with $-1 \leq \zeta \leq 1$. Thus, the Joukowski map squashes the unit circle down to the real line segment $[-1, 1]$. The images of points outside the unit circle fill the rest of the ζ plane, as do the images of the (nonzero) points inside the unit circle. Indeed, if we solve (16.70) for

$$z = \zeta \pm \sqrt{\zeta^2 - 1}, \quad (16.71)$$

we see that every ζ except ± 1 comes from two different points z ; for ζ not on the critical line segment $[-1, 1]$, one point lies inside and one lies outside the unit circle, whereas if $-1 < \zeta < 1$, the points are situated directly above and below it on the circle. Therefore, (16.70) defines a one-to-one conformal map from the exterior of the unit circle $\{|z| > 1\}$ onto the exterior of the unit line segment $\mathbb{C} \setminus [-1, 1]$.

Under the Joukowski map, the concentric circles $|z| = r \neq 1$ are mapped to ellipses with foci at ± 1 in the ζ plane; see Figure 16.20. The effect on circles not centered at the origin is quite interesting. The image curves take on a wide variety of shapes; several examples are plotted in Figure 16.21. If the circle passes through the singular point $z = 1$, then its image is no longer smooth, but has a cusp at $\zeta = 1$; this happens in the last 5

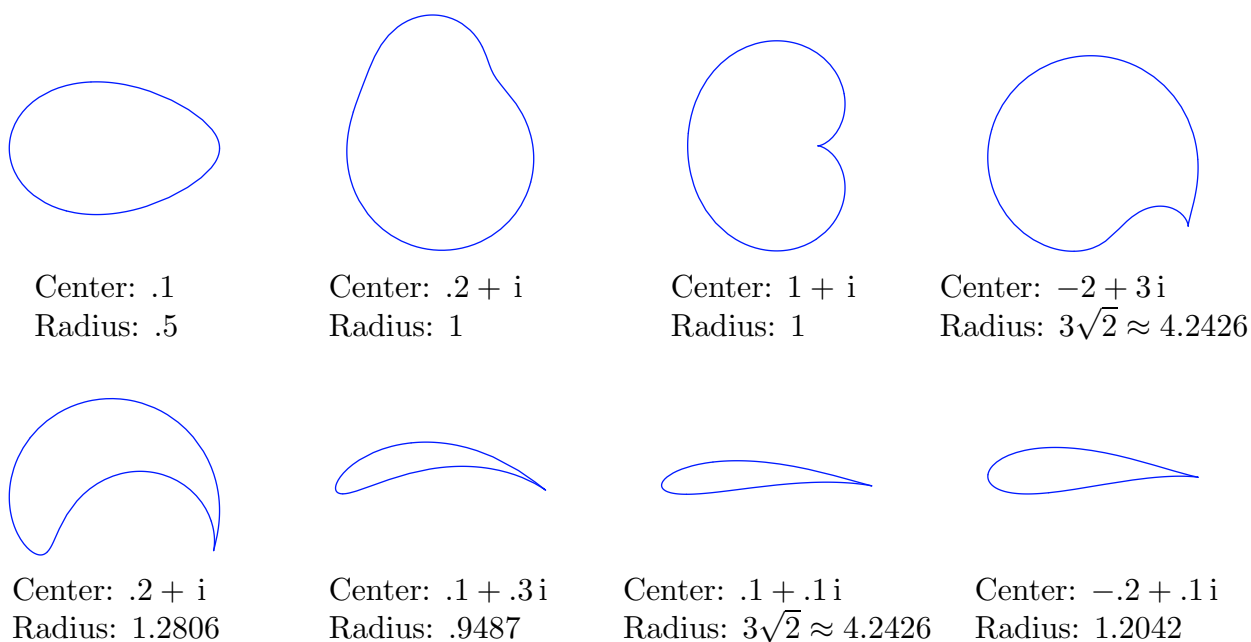


Figure 16.21. Airfoils Obtained from Circles via the Joukowski Map.

of the figures. Some of the image curves have the shape of the cross-section through an airplane wing or *airfoil*. Later we will see how to construct the physical fluid flow around such an airfoil, which proved to be a critical step in early airplane design.

Composition and The Riemann Mapping Theorem

One of the strengths of the method of conformal mapping is that one can build up lots of complicated examples by simply composing elementary mappings. The method rests on the simple fact that the composition of two complex analytic functions is also complex analytic. This is the complex counterpart of the result, learned in first year calculus, that the composition of two differentiable functions is itself differentiable.

Proposition 16.29. *If $w = f(z)$ is an analytic function of the complex variable $z = x + iy$, and $\zeta = g(w)$ is an analytic function of the complex variable $w = u + iv$, then the composition[†] $\zeta = h(z) \equiv g \circ f(z) = g(f(z))$ is an analytic function of z .*

Proof: The proof that the composition of two differentiable functions is differentiable is identical to the real variable version, [9, 167], and need not be reproduced here. The derivative of the composition is explicitly given by the usual chain rule:

$$\frac{d}{dz} g \circ f(z) = g'(f(z)) f'(z), \quad \text{or, in Leibnizian notation,} \quad \frac{d\zeta}{dz} = \frac{d\zeta}{dw} \frac{dw}{dz}. \quad (16.72)$$

Further details are left to the reader.

Q.E.D.

[†] Of course, to properly define the composition, we need to ensure that the range of the function $w = f(z)$ is contained in the domain of the function $\zeta = g(w)$.

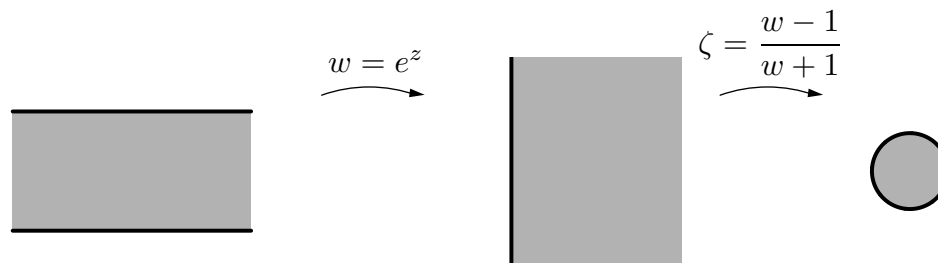


Figure 16.22. Composition of Conformal Maps.

If both f and g are one-to-one, so is the composition $h = g \circ f$. Moreover, the composition of two conformal maps is also conformal, a fact that is immediate from the definition, or by using the chain rule (16.72) to show that

$$h'(z) = g'(f(z)) f'(z) \neq 0 \quad \text{provided} \quad g'(f(z)) \neq 0 \quad \text{and} \quad f'(z) \neq 0.$$

Thus, if f and g satisfy the conformality condition (16.52), so does $h = g \circ f$. *Q.E.D.*

Example 16.30. As we learned in Example 16.20, the exponential function

$$w = e^z$$

maps the horizontal strip $S = \{-\frac{1}{2}\pi < \operatorname{Im} z < \frac{1}{2}\pi\}$ conformally onto the right half plane $R = \{\operatorname{Re} w > 0\}$. On the other hand, Example 16.22 tells us that the linear fractional transformation

$$\zeta = \frac{w-1}{w+1}$$

maps the right half plane R conformally to the unit disk $D = \{|\zeta| < 1\}$. Therefore, the composition

$$\zeta = \frac{e^z - 1}{e^z + 1} \tag{16.73}$$

is a one-to-one conformal map from the horizontal strip S to the unit disk D , as illustrated in Figure 16.22.

Recall that our motivating goal is to use analytic functions/conformal maps to solve boundary value problems for the Laplace equation on a complicated domain Ω by transforming them to boundary value problems on the unit disk. Of course, the key question the student should be asking at this point is: Is there, in fact, a conformal map $\zeta = g(z)$ from a given domain Ω to the unit disk $D = g(\Omega)$? The theoretical answer is the celebrated *Riemann Mapping Theorem*.

Theorem 16.31. *If $\Omega \subsetneq \mathbb{C}$ is any simply connected open subset, not equal to the entire complex plane, then there exists a one-to-one complex analytic map $\zeta = g(z)$, satisfying the conformality condition $g'(z) \neq 0$ for all $z \in \Omega$, that maps Ω to the unit disk $D = \{|\zeta| < 1\}$.*

Thus, *any* simply connected domain — with one exception, the entire complex plane — can be conformally mapped the unit disk. (Exercise ■ provides a reason for this exception.) Note that Ω need *not* be bounded for this to hold. Indeed, the conformal map (16.60)

takes the unbounded right half plane $R = \{ \operatorname{Re} z > 0 \}$ to the unit disk. The proof of this important theorem relies on some more advanced results in complex analysis, and can be found, for instance, in [4].

The Riemann Mapping Theorem guarantees the existence of a conformal map from any simply connected domain to the unit disk, but its proof is not constructive, and so provides no clue as to how to actually construct the desired mapping. And, in general, this is not an easy task. In practice, one assembles a repertoire of useful conformal maps that apply to particular domains of interest. An extensive catalog can be found in [Cmap]. More complicated maps can then be built up by composition of the basic examples. Ultimately, though, the determination of a suitable conformal map is more an art than a systematic science. Numerical methods for constructing conformal maps can be found in [TH].

Let us consider a few additional examples beyond those already encountered:

Example 16.32. Suppose we are asked to conformally map the upper half plane $U = \{ \operatorname{Im} z > 0 \}$ to the unit disk $D = \{ |\zeta| < 1 \}$. We already know that the linear fractional transformation

$$\zeta = g(w) = \frac{w - 1}{w + 1}$$

maps the right half plane $R = \{ \operatorname{Re} z > 0 \}$ to $D = g(R)$. On the other hand, multiplication by $i = e^{i\pi/2}$, with $z = h(w) = iw$, rotates the complex plane by 90° and so maps the right half plane R to the upper half plane $U = h(R)$. Its inverse $h^{-1}(z) = -iz$ will therefore map U to $R = h^{-1}(U)$. Therefore, to map the upper half plane to the unit disk, we compose these two maps, leading to the conformal map

$$\zeta = g \circ h^{-1}(z) = \frac{-iz - 1}{-iz + 1} = \frac{iz + 1}{iz - 1} \quad (16.74)$$

from U to D .

In a similar vein, we already know that the squaring map $w = z^2$ maps the upper right quadrant $Q = \{ 0 < \operatorname{ph} z < \frac{1}{2}\pi \}$ to the upper half plane U . Composing this with our previously constructed map — which requires using w instead of z in the previous formula (16.74) — leads to the conformal map

$$\zeta = \frac{iz^2 + 1}{iz^2 - 1} \quad (16.75)$$

that maps the quadrant Q to the unit disk D .

Example 16.33. The goal of this example is to construct a conformal map that takes a half disk

$$D_+ = \{ |z| < 1, y = \operatorname{Im} z > 0 \} \quad (16.76)$$

to the full unit disk $D = \{ \|\zeta\| < 1 \}$. The answer is *not* $\zeta = z^2$ because the image of D_+ omits the positive real axis, resulting in a disk with a slit cut out of it: $\{ \|\zeta\| < 1, 0 < \operatorname{ph} \zeta < 2\pi \}$. To obtain the entire disk as the image of the conformal map, we must think a little harder. The first observation is that the map $z = (w - 1)/(w + 1)$ that we analyzed in Example 16.22 takes the right half plane $R = \{ \operatorname{Re} w > 0 \}$ to the unit

disk. Moreover, it maps the upper right quadrant $Q = \{ 0 < \text{ph } w < \frac{1}{2} \pi \}$ to the half disk (16.76). Its inverse,

$$w = \frac{z+1}{z-1} \quad (16.77)$$

will therefore map the half disk, $z \in D_+$, to the upper right quadrant $w \in Q$.

On the other hand, we just constructed a conformal map (16.75) that takes the upper right quadrant Q to the unit disk D . Therefore, if compose the two maps (replacing z by w in (16.75) and then using (16.77)), we obtain the desired conformal map:

$$\zeta = \frac{i w^2 + 1}{i w^2 - 1} = \frac{i \left(\frac{z+1}{z-1} \right)^2 + 1}{i \left(\frac{z+1}{z-1} \right)^2 - 1} = \frac{(i+1)(z^2+1) + 2(i-1)z}{(i-1)(z^2+1) + 2(i+1)z}.$$

The formula can be further simplified by multiplying numerator and denominator by $i+1$, and so

$$\zeta = -i \frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}.$$

The leading factor $-i$ is unimportant and can be omitted, since it merely rotates the disk by -90° , and so

$$\zeta = \frac{z^2 + 2iz + 1}{z^2 - 2iz + 1} \quad (16.78)$$

is an equally valid solution to our problem.

Finally, as noted in the preceding example, the conformal map guaranteed by the Riemann Mapping Theorem is *not* unique. Since the linear fractional transformations (16.63) map the unit disk to itself, we can compose them with any conformal Riemann mapping to produce additional conformal maps from a simply-connected domain to the unit disk. For example, composing (16.63) with (16.73) produces a family of mappings

$$\zeta = \frac{1 + e^z - \alpha(1 - e^z)}{\overline{\alpha}(1 + e^z) - 1 + e^z}, \quad (16.79)$$

which, for any $|\alpha| < 1$, maps the strip $S = \{ -\frac{1}{2} \pi < \text{Im } z < \frac{1}{2} \pi \}$ onto the unit disk. With a little more work, it can be shown that this is the only ambiguity, and so, for instance, (16.79) forms a complete list of one-to-one conformal maps from S to D .

Annular Domains

The Riemann Mapping Theorem does not apply to non-simply connected domains. For purely topological reasons, a hole cannot be made to disappear under a one-to-one continuous mapping — much less a conformal map — and so a non-simply connected domain cannot be mapped in a one-to-one manner onto the unit disk. So we must look elsewhere for a simple model domain.

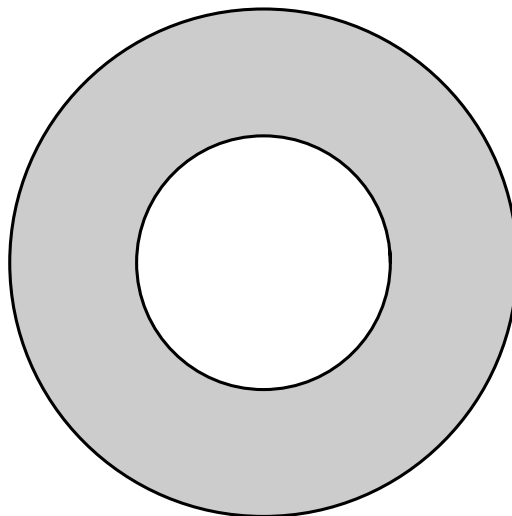


Figure 16.23. An Annulus.

The simplest non-simply connected domain is an *annulus* consisting of the points between two concentric circles

$$A_{r,R} = \{ r < |\zeta| < R \}, \quad (16.80)$$

which, for simplicity, is centered around the origin; see Figure 16.23. The case $r = 0$ corresponds to a punctured disk, while $R = \infty$ gives the exterior of a disk or radius r . It can be proved, [Cmap], that any other domain with a single hole can be mapped to an annulus. The annular radii r, R are not uniquely specified; indeed the linear map $\zeta = \alpha z$ maps the annulus (16.80) to a rescaled annulus $A_{\rho r, \rho R}$ whose inner and outer radii have both been scaled by the factor $\rho = |\alpha|$. But the ratio[†] r/R of the inner to outer radius of the annulus is uniquely specified; annuli with different ratios *cannot* be mapped to each other by a conformal map.

Example 16.34. Let $c > 0$. Consider the domain

$$\Omega = \{ |z| < 1 \quad \text{and} \quad |z - c| > c \}$$

contained between two nonconcentric circles. To keep the computations simple, we take the outer circle to have radius 1 (which can always be arranged by scaling, anyway) while the inner circle has center at the point $z = c$ on the real axis and radius c , which means that it passes through the origin. We must restrict $c < \frac{1}{2}$ in order that the inner circle not overlap with the outer circle. Our goal is to conformally map this non-concentric annular

[†] If $r = 0$ or $R = \infty$, but not both, then $r/R = 0$ by convention. The punctured plane, where $r = 0$ and $R = \infty$ remains a separate case.

domain to a concentric annulus of the form

$$A_{r,1} = \{ r < |\zeta| < 1 \}$$

by a conformal map $\zeta = g(z)$.

Now, according, to Example 16.23, a linear fractional transformation of the form

$$\zeta = g(z) = \frac{z - \alpha}{\overline{\alpha}z - 1} \quad \text{with} \quad |\alpha| < 1 \quad (16.81)$$

maps the unit disk to itself. Moreover, as remarked earlier, and demonstrated in Exercise ■, linear fractional transformations always map circles to circles. Therefore, we seek a particular value of α that maps the inner circle $|z - c| = c$ to a circle of the form $|\zeta| = r$ centered at the origin. We choose α real and try to map the points 0 and $2c$ on the inner circle to the points r and $-r$ on the circle $|\zeta| = r$. This requires

$$g(0) = \alpha = r, \quad g(2c) = \frac{2c - \alpha}{2c\alpha - 1} = -r. \quad (16.82)$$

Substituting the first into the second leads to the quadratic equation

$$c\alpha^2 - \alpha + c = 0.$$

There are two real solutions:

$$\alpha = \frac{1 - \sqrt{1 - 4c^2}}{2c} \quad \text{and} \quad \alpha = \frac{1 + \sqrt{1 - 4c^2}}{2c}. \quad (16.83)$$

Since $0 < c < \frac{1}{2}$, the second solution has $\alpha > 1$, and hence is inadmissible. Therefore, the first solution yields the required conformal map

$$\zeta = \frac{z - 1 + \sqrt{1 - 4c^2}}{(1 - \sqrt{1 - 4c^2})z - 2c}.$$

Note in particular that the radius $r = \alpha$ of the inner circle in $A_{r,1}$ is *not the same* as the radius c of the inner circle in Ω . For example, taking $c = \frac{2}{5}$, equation (16.83) implies $\alpha = \frac{1}{2}$, and hence the linear fractional transformation $\zeta = \frac{2z - 1}{z - 2}$ maps the annular domain $\Omega = \{ |z| < 1, |z - \frac{2}{5}| > \frac{2}{5} \}$ to the concentric annulus $A = A_{1/2,1} = \{ \frac{1}{2} < |\zeta| < 1 \}$. In Figure 16.24, we plot the non-concentric circles in Ω that are mapped to concentric circles in the annulus A .

Applications to Harmonic Functions and Laplace's Equation

Let us now apply our newly acquired expertise in conformal mapping to the study of harmonic functions and boundary value problems for the Laplace equation. Our goal was to change a boundary value problem on a domain Ω into a boundary value problem on the unit disk D that we know how to solve. To this end, suppose we know a conformal map $\zeta = g(z)$ that takes $z \in \Omega$ to $\zeta \in D$. As we know, the real and imaginary parts of an analytic function $F(\zeta)$ defined on D define harmonic functions. Moreover, according

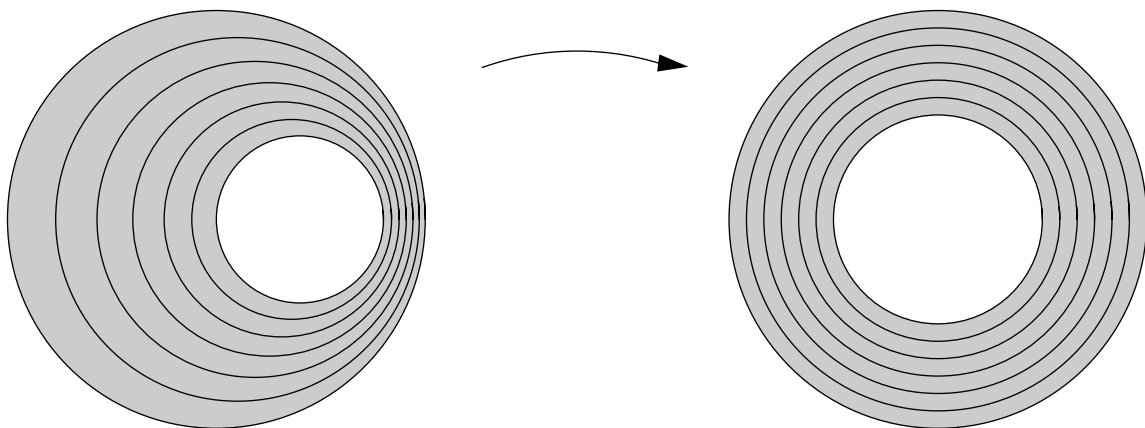


Figure 16.24. Conformal Map for Non-concentric Annulus.

to Proposition 16.29, the composition $f(z) = F(g(z))$ defines an analytic function whose real and imaginary parts are harmonic functions on Ω . Thus, the conformal mapping can be regarded as a change of variables between their harmonic real and imaginary parts. In fact, this property does not even require the harmonic function to be the real part of an analytic function, i.e., we are not required to assume the existence of a harmonic conjugate.

Proposition 16.35. *If $U(\xi, \eta)$ is a harmonic function of ξ, η , and*

$$\zeta = \xi + i\eta = p(x, y) + iq(x, y) = g(z) \quad (16.84)$$

is any analytic function, then the composition

$$u(x, y) = U(p(x, y), q(x, y)) \quad (16.85)$$

is a harmonic function of x, y .

Proof: This is a straightforward application of the chain rule:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial U}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta} \frac{\partial \eta}{\partial x}, & \frac{\partial u}{\partial y} &= \frac{\partial U}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial U}{\partial \eta} \frac{\partial \eta}{\partial y}, \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 U}{\partial \xi^2} \left(\frac{\partial \xi}{\partial x} \right)^2 + 2 \frac{\partial^2 U}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial^2 U}{\partial \eta^2} \left(\frac{\partial \eta}{\partial x} \right)^2 + \frac{\partial U}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial U}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2}, \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 U}{\partial \xi^2} \left(\frac{\partial \xi}{\partial y} \right)^2 + 2 \frac{\partial^2 U}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} + \frac{\partial^2 U}{\partial \eta^2} \left(\frac{\partial \eta}{\partial y} \right)^2 + \frac{\partial U}{\partial \xi} \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial U}{\partial \eta} \frac{\partial^2 \eta}{\partial y^2}. \end{aligned}$$

Using the Cauchy–Riemann equations

$$\frac{\partial \xi}{\partial x} = -\frac{\partial \eta}{\partial y}, \quad \frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x},$$

for the analytic function $\zeta = \xi + i\eta$, we find, after some algebra,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left[\left(\frac{\partial \xi}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial x} \right)^2 \right] \left[\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} \right] = |g'(z)|^2 \Delta U,$$

where $|g'(z)|^2 = (\partial\xi/\partial x)^2 + (\partial\eta/\partial x)^2$. We conclude that whenever $U(\xi, \eta)$ is any harmonic function, and so solves the Laplace equation $\Delta U = 0$ (in the ξ, η variables), then $u(x, y)$ is a solution to the Laplace equation $\Delta u = 0$ in the x, y variables, and is thus also harmonic. *Q.E.D.*

This observation has profound consequences for boundary value problems arising in physical applications. Suppose we wish to solve the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = h \quad \text{on} \quad \partial\Omega, \quad (16.86)$$

on a simply connected domain[†] $\Omega \subsetneq \mathbb{C}$. Let $\zeta = g(z) = p(x, y) + i q(x, y)$ be a one-to-one conformal mapping from the domain Ω to the unit disk D , whose existence is guaranteed by the Riemann Mapping Theorem 16.31. (Although its explicit construction may be much more problematic.) Then the change of variables formula (16.85) will map the harmonic function $u(x, y)$ on Ω to a harmonic function $U(\xi, \eta)$ on D . Moreover, the boundary values of $U = H$ on the unit circle ∂D correspond to those of $u = h$ on $\partial\Omega$ by the same change of variables formula:

$$h(x, y) = H(p(x, y), q(x, y)), \quad \text{for} \quad (x, y) \in \partial\Omega. \quad (16.87)$$

We conclude that $U(\xi, \eta)$ solves the Dirichlet problem

$$\Delta U = 0 \quad \text{in} \quad D, \quad U = H \quad \text{on} \quad \partial D.$$

But we already know how to solve the Dirichlet problem on the unit disk by the Poisson integral formula (15.48)! Then the solution to the original boundary value problem is given by the composition formula $u(x, y) = U(p(x, y), q(x, y))$. Thus, the solution to the Dirichlet problem on a unit disk can be used to solve the Dirichlet problem on more complicated planar domains — provided we know the appropriate conformal map.

Example 16.36. According to Example 16.22, the analytic function

$$\xi + i\eta = \zeta = \frac{z-1}{z+1} = \frac{x^2+y^2-1}{(x+1)^2+y^2} + i \frac{2y}{(x+1)^2+y^2} \quad (16.88)$$

maps the right half plane $R = \{x = \operatorname{Re} z > 0\}$ to the unit disk $D = \{|\zeta| < 1\}$. Proposition 16.35 implies that if $U(\xi, \eta)$ is a harmonic function in the unit disk, then

$$u(x, y) = U\left(\frac{x^2+y^2-1}{(x+1)^2+y^2}, \frac{2y}{(x+1)^2+y^2}\right) \quad (16.89)$$

is a harmonic function on the right half plane. (This can, of course, be checked directly by a rather unpleasant chain rule computation.)

To solve the Dirichlet boundary value problem

$$\Delta u = 0, \quad x > 0, \quad u(0, y) = h(y), \quad (16.90)$$

[†] The Riemann Mapping Theorem 16.31 tells us to exclude the case $\Omega = \mathbb{C}$. Indeed, this case is devoid of boundary conditions, and so the problem does not admit a unique solution.

on the right half plane, we adopt the change of variables (16.88) and use the Poisson integral formula to construct the solution to the transformed Dirichlet problem

$$\Delta U = 0, \quad \xi^2 + \eta^2 < 1, \quad U(\cos \varphi, \sin \varphi) = H(\varphi), \quad (16.91)$$

on the unit disk. The relevant boundary conditions are found as follows. Using the explicit form

$$x + iy = z = \frac{1 + \zeta}{1 - \zeta} = \frac{(1 + \zeta)(1 - \bar{\zeta})}{|1 - \zeta|^2} = \frac{1 + \zeta - \bar{\zeta} - |\zeta|^2}{|1 - \zeta|^2} = \frac{1 - \xi^2 - \eta^2 + 2i\eta}{(\xi - 1)^2 + \eta^2}$$

for the inverse map, we see that the boundary point $\zeta = \xi + i\eta = e^{i\varphi}$ on the unit circle ∂D will correspond to the boundary point

$$iy = \frac{2\eta}{(\xi - 1)^2 + \eta^2} = \frac{2i \sin \varphi}{(\cos \varphi - 1)^2 + \sin^2 \varphi} = i \cot \frac{\varphi}{2} \quad (16.92)$$

on the imaginary axis $\partial R = \{ \operatorname{Re} z = 0 \}$. Thus, the boundary data $h(y)$ on ∂R corresponds to the boundary data

$$H(\varphi) = h\left(\cot \frac{1}{2} \varphi\right)$$

on the unit circle. The Poisson integral formula (15.48) can then be applied to solve (16.91), from which we are able to reconstruct the solution (16.89) to the boundary value problem (16.89) on the half plane.

Let's look at an explicit example. If the boundary data on the imaginary axis is provided by the step function

$$u(0, y) = h(y) \equiv \begin{cases} 1, & y > 0, \\ 0, & y < 0, \end{cases}$$

then the corresponding boundary data on the unit disk is a (periodic) step function

$$H(\varphi) = \begin{cases} 1, & 0 < \varphi < \pi, \\ 0, & \pi < \varphi < 2\pi, \end{cases}$$

that has values $+1$ on the upper semicircle, -1 on the lower semicircle, and jump discontinuities at $\zeta = \pm 1$. According to the Poisson formula (15.48), the solution to the latter boundary value problem is given by

$$U(\xi, \eta) = \frac{1}{2\pi} \int_0^\pi \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\varphi - \phi)} d\phi \quad \text{where} \quad \begin{aligned} \xi &= \rho \cos \varphi, \\ \eta &= \rho \sin \varphi. \end{aligned}$$

$$= \frac{1}{\pi} \left[\tan^{-1} \left(\frac{1 + \rho}{1 - \rho} \cot \frac{\varphi}{2} \right) + \tan^{-1} \left(\frac{1 + \rho}{1 - \rho} \tan \frac{\varphi}{2} \right) \right],$$

Finally, we use (16.89) to construct the solution on the upper half plane, although we shall spare the reader the messy details of the final formula. The result is depicted in Figure zpm1h■.

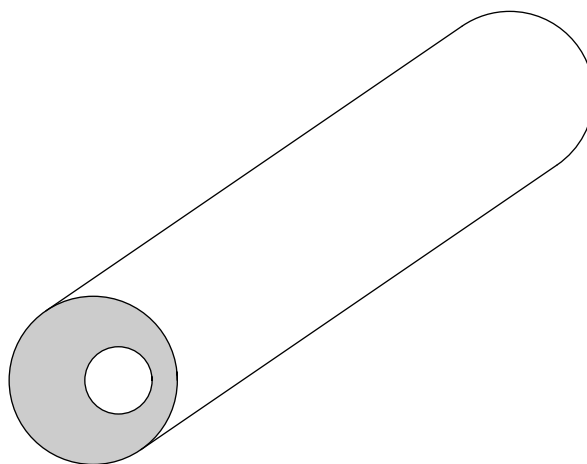


Figure 16.25. Non-Coaxial Cable.

Remark: The solution to the preceding Dirichlet boundary value problem is not, in fact, unique, owing to the unboundedness of the domain. The solution that we pick out by using the conformal map to the unit disk is the one that remains bounded at ∞ . There are other solutions, but they are unbounded as $|z| \rightarrow \infty$ and would correspond to solutions on the unit disk that have some form of delta function singularity in their boundary data at the point -1 ; see Exercise ■.

Example 16.37. *A non-coaxial cable.* The goal of this example is to determine the electrostatic potential inside a non-coaxial cylindrical cable with prescribed constant potential values on the two bounding cylinders, as illustrated in Figure 16.25. Assume for definiteness that the larger cylinder has radius 1, and centered at the origin, while the smaller cylinder has radius $\frac{2}{5}$, and is centered at $z = \frac{2}{5}$. The resulting electrostatic potential will be independent of the longitudinal coordinate, and so can be viewed as a planar potential in the annular domain contained between two circles representing the cross-sections of our cylinders. The desired potential must satisfy the Dirichlet boundary value problem

$$\begin{aligned} \Delta u &= 0 & \text{when} & & |z| < 1 & \text{and} & & \left| z - \frac{2}{5} \right| > \frac{2}{5}, \\ u &= a, & \text{when} & & |z| = 1, & \text{and} & & u = b \quad \text{when} \quad \left| z - \frac{2}{5} \right| = \frac{2}{5}. \end{aligned}$$

According to Example 16.34, the linear fractional transformation

$$\zeta = \frac{2z - 1}{z - 2} \tag{16.93}$$

will map this non-concentric annular domain to the annulus $A_{1/2,1} = \{ \frac{1}{2} < |\zeta| < 1 \}$, which is the cross-section of a coaxial cable. The corresponding transformed potential $U(\xi, \eta)$ has the constant Dirichlet boundary conditions

$$U = a, \quad \text{when} \quad |\zeta| = \frac{1}{2}, \quad \text{and} \quad U = b \quad \text{when} \quad |\zeta| = 1. \tag{16.94}$$

Clearly the coaxial potential U must be a radially symmetric solution to the Laplace equation, and hence, according to (15.64), of the form

$$U(\xi, \eta) = \alpha \log |\zeta| + \beta,$$

for constants α, β . A short computation shows that the particular potential function

$$U(\xi, \eta) = \frac{b-a}{\log 2} \log |\zeta| + b = \frac{b-a}{2 \log 2} \log(\xi^2 + \eta^2) + b$$

satisfies the prescribed boundary conditions (16.94). Therefore, the desired non-coaxial electrostatic potential

$$u(x, y) = \frac{b-a}{\log 2} \log \left| \frac{2z-1}{z-2} \right| + b = \frac{b-a}{2 \log 2} \log \left(\frac{(2x-1)^2 + y^2}{(x-2)^2 + y^2} \right) + b \quad (16.95)$$

is obtained by composition with the conformal map (16.93). The particular case $a = 0$, $b = 1$, is plotted in Figure ncoaxep■.

Remark: The same harmonic function solves the problem of determining the equilibrium temperature in an annular plate whose inner boundary is kept at a temperature $u = a$ while the outer boundary is kept at temperature $u = b$. One could also interpret this solution as the equilibrium temperature of a three-dimensional cylindrical body contained between two non-coaxial cylinders that are held at fixed temperatures. The body's temperature (16.95) will only depend upon the transverse coordinates x, y and not upon the longitudinal coordinate z .

Applications to Fluid Flow

Conformal mappings are particularly useful in the analysis of planar ideal fluid flow. Recall that if $\Theta(\zeta) = \Phi(\xi, \eta) + i\Psi(\xi, \eta)$ is an analytic function that represents the complex potential function for a steady state fluid flow in a planar domain $\zeta \in D$, then we can interpret its real part $\Phi(\xi, \eta)$ as the velocity potential, while the imaginary part $\Psi(\xi, \eta)$ is the harmonic conjugate stream function. The level curves of Φ are the equipotential lines, and, except at stagnation points where $\Theta'(\zeta) = 0$, are orthogonal to the level curves of Ψ , which are the streamlines followed by the individual fluid particles.

Composing the complex potential with a conformal map $\zeta = g(z)$ leads to a transformed complex potential $\Theta(g(z)) = \chi(z) = \varphi(x, y) + i\psi(x, y)$ on the corresponding domain $z \in \Omega$. A key fact is that the conformal map will take isopotential lines of φ to isopotential lines of Φ and streamlines of ψ to streamlines of Ψ . Conformality implies that the orthogonality relations among isopotentials and streamlines away from stagnation points is maintained.

Let us concentrate on the case of flow past a solid object. In three dimensions, the object is assumed to have a uniform shape in the axial direction, and so we can restrict our attention to a planar fluid flow around a closed, bounded subset $D \subset \mathbb{R}^2 \simeq \mathbb{C}$ representing the cross-section of our cylindrical object, as in Figure crss■. The (complex) velocity and potential are defined on the complementary domain $\Omega = \mathbb{C} \setminus D$ occupied by the fluid. The ideal flow assumptions of incompressibility and irrotationality are reasonably accurate if

the flow is laminar, meaning far away from turbulent. The velocity potential $\varphi(x, y)$ will satisfy the Laplace equation $\Delta\varphi = 0$ in the exterior domain Ω . For a solid object, we should impose the homogeneous Neumann boundary conditions

$$\frac{\partial\varphi}{\partial\mathbf{n}} = 0 \quad \text{on the boundary} \quad \partial\Omega = \partial D, \quad (16.96)$$

indicating that there no fluid flux into the object. We note that, according to Exercise ■, a conformal map will automatically preserve the Neumann boundary conditions.

In addition, since the flow is taking place on an unbounded domain, we need to specify the fluid motion at large distances. We shall assume our object is placed in a uniform horizontal flow, as in Figure hflow■. Thus, far away, the object will not affect the flow, and so the velocity should approximate the uniform velocity field $\mathbf{v} = (1, 0)^T$, where, for simplicity, we choose our physical units so that the asymptotic speed of the fluid is equal to 1. Equivalently, the velocity potential should satisfy

$$\varphi(x, y) \approx x, \quad \text{so} \quad \nabla\varphi \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{when} \quad x^2 + y^2 \gg 0.$$

Remark: An alternative physical interpretation is that the fluid is at rest, while the object moves through the fluid at unit speed 1 in a horizontal direction. For example, think of an airplane flying through the air at constant speed. If we adopt a moving coordinate system by sitting inside the airplane, then the effect is as if the object is sitting still while the air is moving towards us at unit speed.

Example 16.38. The simplest example is a flat plate moving through the fluid in a horizontal direction. The plate's cross-section is a horizontal line segment, and, for simplicity, we take it to be the segment $D = [-1, 1]$ lying on the real axis. If the plate is very thin, it will have absolutely no effect on the horizontal flow of the fluid, and, indeed, the velocity potential is given by

$$\varphi(x, y) = x, \quad \text{for} \quad x + iy \in \Omega = \mathbb{C} \setminus [-1, 1].$$

Note that $\nabla\varphi = (1, 0)^T$, and hence this flow satisfies the Neumann boundary conditions (16.96) on the horizontal segment $D = \partial\Omega$. The corresponding complex potential is $\chi(z) = z$, with complex velocity $f(z) = \chi'(z) = 1$.

Example 16.39. Recall that the Joukowski conformal map defined by the analytic function

$$\zeta = g(z) = \frac{1}{2} \left(z + \frac{1}{z} \right) \quad (16.97)$$

squashes the unit circle $|z| = 1$ down to the real line segment $[-1, 1]$ in the ζ plane. Therefore, it will map the fluid flow outside the unit disk (which is the cross-section of a circular cylinder) to the fluid flow past the line segment, which, according to the previous example, has complex potential $\Theta(\zeta) = \zeta$. As a result, the complex potential for the flow past a disk is the Joukowski function

$$\chi(z) = \Theta \circ g(z) = g(z) = \frac{1}{2} \left(z + \frac{1}{z} \right). \quad (16.98)$$

Except for a factor of $\frac{1}{2}$, this agrees with the flow potential we derived in Example 16.17. The difference is that, at large distances, the potential

$$\chi(z) \approx \frac{1}{2}z \quad \text{for} \quad |z| \gg 1.$$

corresponds to uniform horizontal flow whose velocity $(\frac{1}{2}, 0)^T$ is half as fast. The discrepancy between the two flows can easily be rectified by multiplying (16.98) by 2, whose only effect is to speed up the flow.

Example 16.40. Let us next consider the case of a tilted plate in a uniformly horizontal fluid flow. Thus, the cross-section is the line segment

$$z(t) = t e^{i\theta}, \quad -1 \leq t \leq 1,$$

obtained by rotating the horizontal line segment $[-1, 1]$ through an angle θ , as in Figure tilt. The goal is to construct a fluid flow past the tilted segment that is asymptotically horizontal at large distance.

The critical observation is that, while the effect of rotating a plate in a fluid flow is not so evident, rotating a circularly symmetric disk has no effect on in the flow around it. Thus, the rotation $w = e^{-i\theta} z$ maps the Joukowski potential (16.98) to the complex potential

$$\Upsilon(w) = \chi(e^{i\theta} w) = \frac{1}{2} \left(e^{i\theta} w + \frac{e^{-i\theta}}{w} \right).$$

The streamlines of the induced flow are no longer asymptotically horizontal, but rather at an angle $-\theta$. If we now apply the original Joukowski map (16.97) to the rotated flow, the circle is again squashed down to the horizontal line segment, but the flow lines continue to be at angle $-\theta$ at large distances. Thus, if we then rotate the resulting flow through an angle θ , the net effect will be to tilt the segment to the desired angle θ while rotating the streamlines to be asymptotically horizontal. Putting the pieces together, we have the final complex potential in the form

$$\chi(z) = e^{i\theta} \left(z \cos \theta - i \sin \theta \sqrt{z^2 - e^{-2i\theta}} \right). \quad (16.99)$$

Sample streamlines for the flow at several attack angles are plotted in Figure tilt.

Example 16.41. As we discovered in Example 16.28, applying the Joukowski map to off-center disks will, in favorable configurations, produce airfoil-shaped objects. The fluid motion around such airfoils can thus be obtained by applying the Joukowski map to the flow past such an off-center circle.

First, an affine map

$$w = \alpha z + \beta$$

has the effect of moving the unit disk $|z| \leq 1$ to the disk $|w - \beta| \leq |\alpha|$ with center β and radius $|\alpha|$. In particular, the boundary circle will continue to pass through the point $w = 1$ provided $|\alpha| = |1 - \beta|$. Moreover, as noted in Example 16.18, the angular component of α has the effect of a rotation, and so the streamlines around the new disk

will, asymptotically, be at an angle $\varphi = \text{ph } \alpha$ with the horizontal. We then apply the Joukowski transformation

$$\zeta = \frac{1}{2} \left(w + \frac{1}{w} \right) = \frac{1}{2} \left(\alpha z + \beta + \frac{1}{\alpha z + \beta} \right) \quad (16.100)$$

to map the disk to the airfoil shape. The resulting complex potential for the flow past the airfoil is obtained by substituting the inverse map

$$z = \frac{w - \beta}{\alpha} = \frac{\zeta - \beta + \sqrt{\zeta^2 - 1}}{\alpha},$$

into the original potential (16.98), whereby

$$\Theta(\zeta) = \frac{1}{2} \left(\frac{\zeta - \beta + \sqrt{\zeta^2 - 1}}{\alpha} + \frac{\alpha(\zeta - \beta - \sqrt{\zeta^2 - 1})}{\beta^2 + 1 - 2\beta\zeta} \right).$$

Since the streamlines have been rotated through an angle $\varphi = \text{ph } \alpha$, we then rotate the final result back by multiplying by $e^{i\varphi}$ in order to see the effect of the airfoil tilted at an angle $-\varphi$ in a horizontal flow. Sample streamlines are graphed in Figure [airfoillift](#).

We can interpret all these examples as planar cross-sections of three-dimensional fluid flows past an airplane wing oriented in the longitudinal z direction. The wing is assumed to have a uniform cross-section shape, and the flow not dependent upon the axial z coordinate. For sufficiently long wings flying in laminar (non-turbulent) flows, this model will be valid away from the wing tips. Understanding the dynamics of more complicated airfoils with varying cross-section and/or faster motion requires a fully three-dimensional fluid model. For such problems, complex analysis is no longer applicable, and, for the most part, one must rely on large scale numerical integration. Only in recent years have computers become sufficiently powerful to compute realistic three-dimensional fluid motions — and then only in reasonably “mild” scenarios[†]. The two-dimensional versions that have been analyzed here still provide important clues to the behavior of a three-dimensional flow, as well as useful initial approximations to the three-dimensional airplane wing design problem.

Unfortunately, there is a major flaw with the airfoils that we have just designed. As we will see, potential flows do not produce any lift, and hence such airplanes would not fly. Fortunately for us, the physical flow is not of this nature! In order to understand how lift enters into the picture, we need to study complex integration, and so we will return to this example later. In Example 16.56, we shall construct an alternative flow past an airfoil that continues to have the correct asymptotic behavior at large distances, while inducing a nonzero lift on the wing. This is the secret to flight.

Poisson's Equation and the Green's Function

Although designed for solving the homogeneous Laplace equation, the method of conformal mapping can also be used to solve its inhomogeneous counterpart — the Poisson

[†] The definition of mild relies on the magnitude of the Reynolds number, [14], an overall measure of the flow's complexity.

equation. As we learned in Chapter 15, to solve an inhomogeneous boundary value problem it suffices to solve the problem when the right hand side is a delta function concentrated at a point in the domain:

$$-\Delta u = \delta_\zeta(x, y) = \delta(x - \xi) \delta(y - \eta), \quad \zeta = \xi + i\eta \in \Omega,$$

subject to homogeneous boundary conditions (Dirichlet or mixed) on $\partial\Omega$. (As usual, we exclude pure Neumann boundary conditions due to lack of existence/uniqueness.) The solution

$$u(x, y) = G_\zeta(x, y) = G(x, y; \xi, \eta)$$

is the *Green's function* for the given boundary value problem. With the Green's function in hand, the solution to the homogeneous boundary value problem under a general external forcing,

$$-\Delta u = f(x, y),$$

is then provided by the superposition principle

$$u(x, y) = \iint_{\Omega} G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta. \quad (16.101)$$

For the planar Poisson equation, the starting point is the logarithmic potential function

$$u(x, y) = \operatorname{Re} \frac{1}{2\pi} \log z = \frac{1}{2\pi} \log |z| = \frac{1}{4\pi} \log(x^2 + y^2), \quad (16.102)$$

which solves the Dirichlet problem

$$-\Delta u = \delta_0(x, y), \quad (x, y) \in D, \quad u = 0 \quad \text{on} \quad \partial D,$$

on the unit disk D for an impulse concentrated at the origin; see Section 15.3 for details. How do we obtain the corresponding solution when the unit impulse is concentrated at another point $\zeta = \xi + i\eta \in D$ instead of the origin? According to Example 16.23, the linear fractional transformation

$$w = g(z) = \frac{z - \zeta}{\bar{\zeta} z - 1}, \quad \text{where} \quad |\zeta| < 1, \quad (16.103)$$

maps the unit disk to itself, moving the point $z = \zeta$ to the origin $w = g(\zeta) = 0$. The logarithmic potential $U = \frac{1}{2\pi} \log |w|$ will thus be mapped to the Green's function

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \log \left| \frac{z - \zeta}{\bar{\zeta} z - 1} \right| \quad (16.104)$$

at the point $\zeta = \xi + i\eta$. Indeed, by the properties of conformal mapping, since U is harmonic except at the singularity $w = 0$, the function (16.104) will also be harmonic except at the image point $z = \zeta$. The fact that the mapping does not affect the delta function singularity is not hard to check; the details are relegated to Exercise ■. Moreover,

since the conformal map does not alter the boundary $|z| = 1$, the function (16.104) continues to satisfy the homogeneous Dirichlet boundary conditions.

Formula (16.104) reproduces the Poisson formula (15.78) for the Green's function that we previously derived by the method of images. This identification can be verified by substituting $z = r e^{i\theta}$, $\zeta = \rho e^{i\varphi}$, or, more simply, by noting that the numerator in the logarithmic fraction gives the potential due to a unit impulse at $z = \zeta$, while the denominator represents the image potential at $z = 1/\bar{\zeta}$ required to cancel out the effect of the interior potential on the boundary of the unit disk.

Now that we know the Green's function on the unit disk, we can use the methods of conformal mapping to produce the Green's function for any other simply connected domain $\Omega \subsetneq \mathbb{C}$.

Proposition 16.42. *Let $w = g(z)$ denote the conformal map that takes the domain $z \in \Omega$ to the unit disk $w \in D$, guaranteed by the Riemann Mapping Theorem 16.31. Then the Green's function associated with homogeneous Dirichlet boundary conditions on Ω is explicitly given by*

$$G(z; \zeta) = \frac{1}{2\pi} \log \left| \frac{g(z) - g(\zeta)}{g(\zeta)g(z) - 1} \right|. \quad (16.105)$$

Example 16.43. According to Example 16.22, the analytic function

$$w = \frac{z - 1}{z + 1}$$

maps the right half plane $x = \operatorname{Re} z > 0$ to the unit disk $|\zeta| < 1$. Therefore, by (16.105), the Green's function for the right half plane has the form

$$G(z; \zeta) = \frac{1}{2\pi} \log \left| \frac{\frac{z-1}{z+1} - \frac{\zeta-1}{\zeta+1}}{\frac{z-1}{z+1} \frac{\bar{\zeta}-1}{\bar{\zeta}+1} - 1} \right| = \frac{1}{2\pi} \log \left| \frac{(\bar{\zeta}+1)(z-\zeta)}{(z+1)(z-\bar{\zeta})} \right|. \quad (16.106)$$

One can then write the solution to the Poisson equation in a superposition as in (16.101).

16.5. Complex Integration.

The magic and power of calculus ultimately rests on the amazing fact that differentiation and integration are mutually inverse operations. And, just as complex functions have many remarkable differentiability properties not enjoyed by their real siblings, so the sublime beauty and innate structure complex integration goes far beyond its more mundane real counterpart. In the remaining two sections of this chapter, we shall develop the basics of complex integration theory and present a few of its important applications.

The first step is to motivate the definition of a complex integral. As you know, the (definite) integral of a real function, $\int_a^b f(t) dt$, is evaluated on an interval $[a, b] \subset \mathbb{R}$. In complex function theory, integrals are taken along curves in the complex plane, and

thus have the flavor of the line integrals appearing in real vector calculus. Indeed, the identification of a complex number $z = x + iy$ with a planar vector $\mathbf{x} = (x, y)^T$ will serve to connect the two theories.

Consider a curve C in the complex plane, parametrized, as in (16.65), by $z(t) = x(t) + iy(t)$ for $a \leq t \leq b$. We define the *integral* of the complex function $f(z)$ along the curve C to be the complex number

$$\int_C f(z) dz = \int_a^b f(z(t)) \frac{dz}{dt} dt. \quad (16.107)$$

We shall always assume that the integrand $f(z)$ is a well-defined complex function at each point on the curve. Let us write out the integrand

$$f(z) = u(x, y) + i v(x, y)$$

in terms of its real and imaginary parts. Also,

$$dz = \frac{dz}{dt} dt = \left(\frac{dx}{dt} + i \frac{dy}{dt} \right) dt = dx + i dy.$$

As a result, the complex integral (16.107) splits up into a pair of real line integrals:

$$\int_C f(z) dz = \int_C (u + i v)(dx + i dy) = \int_C (u dx - v dy) + i \int_C (v dx + u dy). \quad (16.108)$$

Example 16.44. Let us compute complex integrals

$$\int_C z^n dz, \quad (16.109)$$

of the monomial function $f(z) = z^n$, where n is an integer, along several different curves. We begin with a straight line segment along the real axis connecting the points -1 to 1 , which we parametrize by $z(t) = t$ for $-1 \leq t \leq 1$. The defining formula (16.107) implies that the complex integral (16.109) reduces to a real integral:

$$\int_C z^n dz = \int_{-1}^1 t^n dt = \begin{cases} 0, & n = 2k + 1 > 0 \text{ is odd} \\ \frac{2}{n+1}, & n = 2k \geq 0 \text{ is even.} \end{cases},$$

If $n \leq -1$ is negative, then the singularity of the integrand at the origin implies that the integral diverges, and so the complex integral is not defined.

Let us evaluate the same complex integral, but now along a parabolic arc P parametrized by

$$z(t) = t + i(t^2 - 1), \quad -1 \leq t \leq 1.$$

Note that, as graphed in Figure 16.26, the parabola connects the same two points. We again refer back to the basic definition (16.107) to evaluate the integral, so

$$\int_P z^n dz = \int_{-1}^1 [t + i(t^2 - 1)]^n (1 + 2it) dt.$$

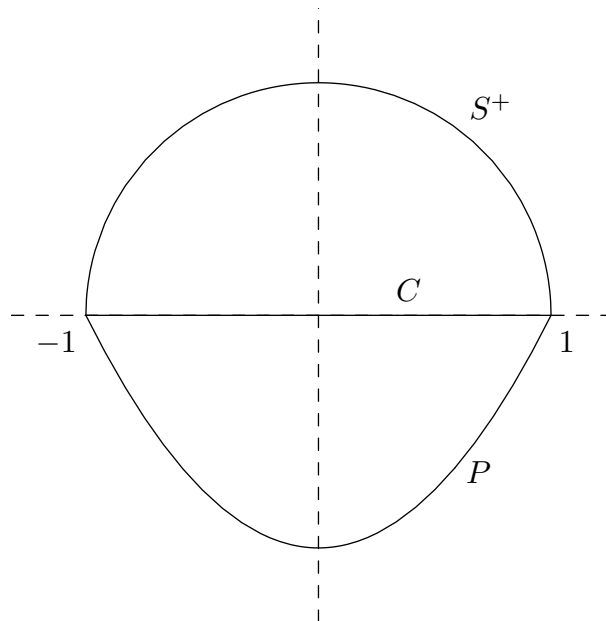


Figure 16.26. Curves for Complex Integration.

We could, at this point, expand the resulting complex polynomial integrand, and then integrate term by term. A more elegant approach is to recognize that the integrand is an exact derivative; namely, by the chain rule

$$\frac{d}{dt} \frac{[t + i(t^2 - 1)]^{n+1}}{n+1} = [t + i(t^2 - 1)]^n (1 + 2it),$$

as long as $n \neq -1$. Therefore, we can use the Fundamental Theorem of Calculus (which works equally well for real integrals of complex-valued functions), to evaluate

$$\int_P z^n dz = \frac{[t + i(t^2 - 1)]^{n+1}}{n+1} \Big|_{t=-1}^1 = \begin{cases} 0, & -1 \neq n = 2k+1 \text{ odd,} \\ \frac{2}{n+1}, & n = 2k \text{ even.} \end{cases}$$

Thus, when $n \geq 0$ is a positive integer, we obtain the same result as before. Interestingly, in this case the complex integral is well-defined even when n is a negative integer because, unlike the real line segment, the parabolic path does *not* go through the singularity of z^n at $z = 0$. The case $n = -1$ needs to be done slightly differently, and integration of $1/z$ along the parabolic path is left as an exercise for the reader — one that requires some care. We recommend trying the exercise now, and then verifying your answer once we have become a little more familiar with basic complex integration techniques.

Finally, let us try integrating around a semi-circular arc, again with the same endpoints

-1 and 1 . If we parametrize the semi-circle S^+ by $z(t) = e^{it}$, $0 \leq t \leq \pi$, we find

$$\begin{aligned} \int_{S^+} z^n dz &= \int_0^\pi z^n \frac{dz}{dt} dt = \int_0^\pi e^{int} i e^{it} dt = \int_0^\pi i e^{i(n+1)t} dt \\ &= \frac{e^{i(n+1)t}}{n+1} \Big|_{t=0}^\pi = \frac{1 - e^{i(n+1)\pi}}{n+1} = \begin{cases} 0, & -1 \neq n = 2k+1 \text{ odd}, \\ -\frac{2}{n+1}, & n = 2k \text{ even}. \end{cases} \end{aligned}$$

This value is the negative of the previous cases — but this can be explained by the fact that the circular arc is oriented to go *from* 1 *to* -1 whereas the line segment and parabola both go *from* -1 *to* 1 . Just as with line integrals, the direction of the curve determines the sign of the complex integral; if we reverse direction, replacing t by $-t$, we end up with the same value as the preceding two complex integrals. Moreover — again provided $n \neq -1$ — it does not matter whether we use the upper semicircle or lower semicircle to go from -1 to 1 — the result is exactly the same. However, the case $n = -1$ is an exception to this “rule”. Integrating along the upper semicircle S^+ from 1 to -1 yields

$$\int_{S^+} \frac{dz}{z} = \int_0^\pi i dt = \pi i, \quad (16.110)$$

whereas integrating along the lower semicircle S^- from 1 to -1 yields the negative

$$\int_{S^-} \frac{dz}{z} = \int_0^{-\pi} i dt = -\pi i. \quad (16.111)$$

Hence, when integrating the function $1/z$, it makes a difference which direction we go around the origin.

Integrating z^n for any integer $n \neq -1$ around an entire circle gives zero — irrespective of the radius. This can be seen as follows. We parametrize a circle of radius r by $z(t) = re^{it}$ for $0 \leq t \leq 2\pi$. Then, by the same computation,

$$\oint_C z^n dz = \int_0^{2\pi} (r^n e^{int})(r i e^{it}) dt = \int_0^{2\pi} i r^{n+1} e^{i(n+1)t} dt = \frac{r^{n+1}}{n+1} e^{i(n+1)t} \Big|_{t=0}^{2\pi} = 0, \quad (16.112)$$

provided $n \neq -1$. Here, as in Appendix A, the circle on the integral sign serves to remind us that we are integrating around a closed curve. The case $n = -1$ remains special. Integrating once around the circle in the counter-clockwise direction yields a nonzero result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} i dt = 2\pi i. \quad (16.113)$$

Let us note that a complex integral does not depend on the particular parametrization of the curve C . It does, however, depend upon the orientation of the curve: if we traverse the curve in the reverse direction, then the complex integral changes its sign:

$$\int_{-C} f(z) dz = - \int_C f(z) dz. \quad (16.114)$$

Moreover, if we chop up the curve into two non-overlapping pieces, $C = C_1 \cup C_2$, with a common orientation, then the complex integral can be decomposed into a sum over the pieces:

$$\int_{C_1 \cup C_2} f(z) = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz. \quad (16.115)$$

For instance, the integral (16.113) of $1/z$ around the circle is the difference of the individual semicircular integrals (16.110–111); the lower semicircular integral acquires a negative sign to flip its orientation so as to agree with that of the entire circle. All these facts are immediate consequences of the basic properties of line integrals, or can be easily proved directly from the defining formula (16.107).

Note: In complex integration theory, a simple closed curve is often referred to as a *contour*, and so complex integration is sometimes referred to as *contour integration*. Unless explicitly stated, we always go around contours in the *counter-clockwise* direction.

Further experiments lead us to suspect that complex integrals are usually path-independent, and hence evaluate to zero around closed contours. One must be careful, though, as the integral (16.113) makes clear. Path independence, in fact, follows from the complex version of the Fundamental Theorem of Calculus.

Theorem 16.45. *Let $f(z) = F'(z)$ be the derivative of a single-valued complex function $F(z)$ defined on a domain $\Omega \subset \mathbb{C}$. Let $C \subset \Omega$ be any curve with initial point α and final point β . Then*

$$\int_C f(z) dz = \int_C F'(z) dz = F(\beta) - F(\alpha). \quad (16.116)$$

Proof: This follows immediately from the definition (16.107) and the chain rule:

$$\int_C F'(z) dz = \int_a^b F'(z(t)) \frac{dz}{dt} dt = \int_a^b \frac{d}{dt} F(z(t)) dt = F(z(b)) - F(z(a)) = F(\beta) - F(\alpha),$$

where $\alpha = z(a)$ and $\beta = z(b)$ are the endpoints of the curve. *Q.E.D.*

For example, when $n \neq -1$, the function $f(z) = z^n$ is the derivative of the single-valued function $F(z) = \frac{1}{n+1} z^{n+1}$. Hence

$$\int_C z^n dz = \frac{\beta^{n+1}}{n+1} - \frac{\alpha^{n+1}}{n+1}$$

whenever C is a curve connecting α to β . When $n < 0$, the curve is not allowed to pass through the origin, $z = 0$, as it is a singularity for z^n .

In contrast, the function $f(z) = 1/z$ is the derivative of the complex logarithm

$$\log z = \log |z| + i \operatorname{ph} z,$$

which is *not* single-valued on all of $\mathbb{C} \setminus \{0\}$, and so Theorem 16.45 cannot be applied directly. However, if our curve is contained within a simply connected subdomain that

does not include the origin, $0 \notin \Omega \subset \mathbb{C}$, then we can use *any* single-valued branch of the complex logarithm to evaluate the integral

$$\int_C \frac{dz}{z} = \log \beta - \log \alpha,$$

where α, β are the endpoints of the curve. Since the common multiples of $2\pi i$ cancel, the answer does not depend upon which particular branch of the complex logarithm is chosen as long as we are consistent in our choice. For example, on the upper semicircle S^+ of radius 1 going from 1 to -1 ,

$$\int_{S^+} \frac{dz}{z} = \log(-1) - \log 1 = \pi i,$$

where we use the branch of $\log z = \log |z| + i \operatorname{ph} z$ with $0 \leq \operatorname{ph} z \leq \pi$. On the other hand, if we integrate on the lower semi-circle S^- going from 1 to -1 , we need to adopt a different branch, say that with $-\pi \leq \operatorname{ph} z \leq 0$. With this choice, the integral becomes

$$\int_{S^-} \frac{dz}{z} = \log(-1) - \log 1 = -\pi i,$$

thus reproducing (16.110, 111). Pay particular attention to the different values of $\log(-1)$ in the two cases!

The most important consequence of Theorem 16.45 is that, as long as the integrand $f(z)$ has a single-valued anti-derivative, its complex integral is independent of the path connecting two points — the value only depends on the endpoints of the curve and not how one gets from point α to point β .

Theorem 16.46. *Let $f(z) = F'(z)$, where $F(z)$ is a single-valued complex function for $z \in \Omega$. If $C \subset \Omega$ is any closed curve, then*

$$\oint_C f(z) dz = 0. \quad (16.117)$$

Conversely, if (16.117) holds for all closed curves $C \subset \Omega$ contained in the domain of definition of $f(z)$, then f admits a single-valued complex anti-derivative with $F'(z) = f(z)$.

Proof: We have already demonstrated the first statement. As for the second, we define

$$F(z) = \int_{z_0}^z f(z) dz,$$

where $z_0 \in \Omega$ is any fixed point, and we choose any convenient curve $C \subset \Omega$ connecting[†] z_0 to z . (16.117) assures us that the value does not depend on the chosen path. The proof that this formula does define an anti-derivative of f is left as an exercise, which can be solved in the same fashion as the case of a real line integral, cf. (22.42). *Q.E.D.*

[†] This assumes Ω is a connected domain; otherwise, apply the result to its individual connected components.

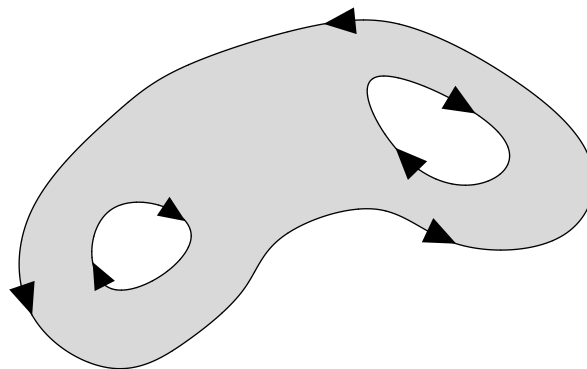


Figure 16.27. Orientation of Domain Boundary.

The preceding considerations suggest the following fundamental theorem, due in its general form to Cauchy. Before stating it, we introduce the convention that a complex function $f(z)$ is to be called *analytic on a domain* $\Omega \subset \mathbb{C}$ provided it is analytic at every point inside Ω and, in addition, remains (at least) continuous on the boundary $\partial\Omega$. When Ω is bounded, its boundary $\partial\Omega$ consists of one or more simple closed curves. In general, as in Green's Theorem A.26, we orient $\partial\Omega$ so that the domain is always on our left hand side. This means that the outermost boundary curve is traversed in the counter-clockwise direction, but any interior holes are take on a clockwise orientation. Our convention is depicted in Figure 16.27.

Theorem 16.47. *If $f(z)$ is analytic on a bounded domain $\Omega \subset \mathbb{C}$, then*

$$\oint_{\partial\Omega} f(z) dz = 0. \quad (16.118)$$

Proof: If we apply Green's Theorem to the two real line integrals in (16.108), we find

$$\oint_{\partial\Omega} u dx - v dy = \iint_{\Omega} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0, \quad \oint_{\partial\Omega} v dx + u dy = \iint_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) = 0,$$

both of which vanish by virtue of the Cauchy–Riemann equations (16.22). *Q.E.D.*

If the domain of definition of our complex function $f(z)$ is simply connected, then, by definition, the interior of any closed curve $C \subset \Omega$ is contained in Ω , and hence Cauchy's Theorem 16.47 implies path independence of the complex integral within Ω .

Corollary 16.48. *If $f(z)$ is analytic on a simply connected domain $\Omega \subset \mathbb{C}$, then its complex integral $\int_C f(z) dz$ for $C \subset \Omega$ is independent of path. In particular,*

$$\oint_C f(z) dz = 0 \quad (16.119)$$

for any closed curve $C \subset \Omega$.

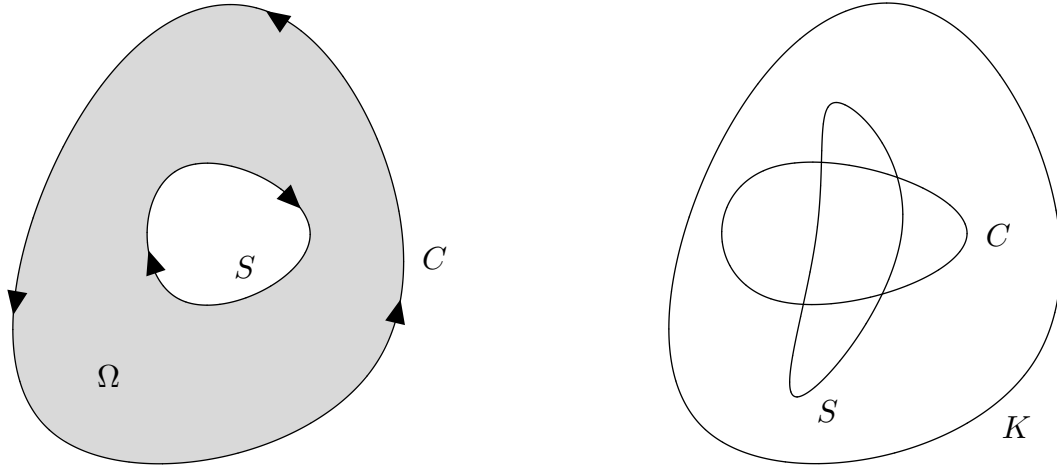


Figure 16.28. Integration Around Two Closed Curves.

Remark: Simple connectivity of the domain is an essential hypothesis — our evaluation (16.113) of the integral of $1/z$ around the unit circle provides a simple counterexample to (16.119) in the non-simply connected domain $\Omega = \mathbb{C} \setminus \{0\}$. Interestingly, this result also admits a converse: a continuous function $f(z)$ that satisfies (16.119) for *all* closed curves is necessarily analytic; see [4] for a proof.

We will also require a slight generalization of this result.

Proposition 16.49. *If $f(z)$ is analytic in a domain that contains two simple closed curves S and C , and the entire region lying between them, then, assuming they are oriented in the same direction,*

$$\oint_C f(z) dz = \oint_S f(z) dz. \quad (16.120)$$

Proof: If C and S do not cross each other, we let Ω denote the domain contained between them, so that $\partial\Omega = C \cup S$; see the first plot in Figure 16.28. According to Cauchy's Theorem 16.47, $\oint_{\partial\Omega} f(z) = 0$. Now, our orientation convention for $\partial\Omega$ means that the outer curve, say C , is traversed in the counter-clockwise direction, while the inner curve S has the opposite, clockwise orientation. Therefore, if we assign both curves the same counter-clockwise orientation,

$$0 = \oint_{\partial\Omega} f(z) = \oint_C f(z) dz - \oint_S f(z) dz,$$

proving (16.120).

If the two curves cross, we can construct a nearby curve $K \subset \Omega$ that neither crosses, as in the second sketch in Figure 16.28. By the preceding paragraph, each integral is equal to that over the third curve,

$$\oint_C f(z) dz = \oint_K f(z) dz = \oint_S f(z) dz,$$

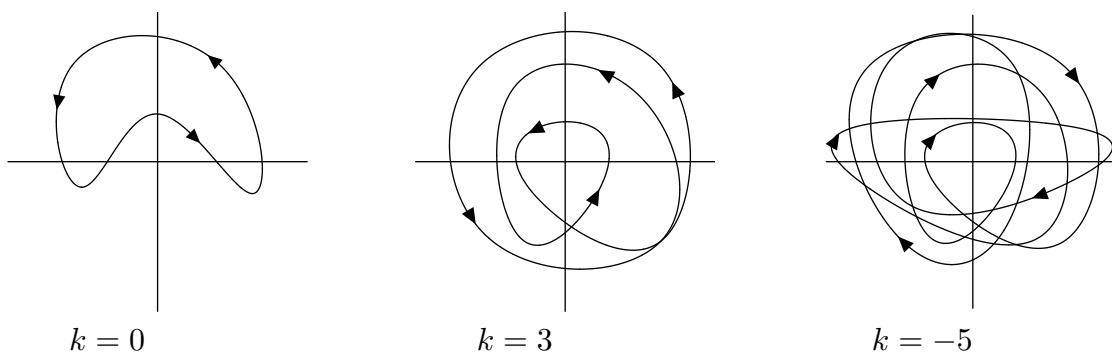


Figure 16.29. Winding Numbers.

and formula (16.120) remains valid.

Q.E.D.

Example 16.50. Consider the function $f(z) = z^n$ where n is an integer[†]. In (16.112), we already computed

$$\oint_C z^n dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1, \end{cases} \quad (16.121)$$

when C is a circle centered at $z = 0$. When $n \geq 0$, Theorem 16.45 immediately implies that the integral of z^n is 0 over *any* closed curve in the plane. The same applies in the cases $n \leq -2$ provided the curve does not pass through the singular point $z = 0$. In particular, the integral is zero around closed curves encircling the origin, even though z^n for $n \leq -2$ has a singularity inside the curve and so Cauchy's Theorem 16.47 does not apply as stated.

The case $n = -1$ has particular significance. Here, Proposition 16.49 implies that the integral is the same as the integral around a circle — provided the curve C also goes once around the origin in a counter-clockwise direction. Thus (16.113) holds for any closed curve that goes counter-clockwise once around the origin. More generally, if the curve goes several times around the origin[‡], then

$$\oint_C \frac{dz}{z} = 2k\pi i \quad (16.122)$$

is an integer multiple of $2\pi i$. The integer k is called the *winding number* of the curve C , and measures the total number of times C goes around the origin. For instance, if C winds three times around 0 in a counter-clockwise fashion, then $k = 3$, while $k = -5$ indicates that the curve winds 5 times around 0 in a clockwise direction, as in Figure 16.29. In particular, a winding number $k = 0$ indicates that C is not wrapped around the origin. If C represents a loop of string wrapped around a pole (the *pole* of $1/z$ at 0) then a winding

[†] When n is fractional or irrational, the integrals are not well-defined owing to the branch point at the origin.

[‡] Such a curve is undoubtedly not simple and must necessarily cross over itself.

number $k = 0$ would indicate that the string can be disentangled from the pole without cutting; nonzero winding numbers would indicate that the string is truly entangled[§].

Lemma 16.51. *If C is any simple closed curve, and a is any point not lying on C , then*

$$\oint_C \frac{dz}{z-a} = \begin{cases} 2\pi i, & a \text{ inside } C \\ 0 & a \text{ outside } C. \end{cases} \quad (16.123)$$

If $a \in C$, then the integral does not converge.

Proof: Note that the integrand $f(z) = 1/(z-a)$ is analytic everywhere except at $z = a$, where it has a simple pole. If a is outside C , then Cauchy's Theorem 16.47 applies, and the integral is zero. On the other hand, if a is inside C , then Proposition 16.49 implies that the integral is equal to the integral around a circle centered at $z = a$. The latter integral can be computed directly by using the parametrization $z(t) = a + r e^{it}$ for $0 \leq t \leq 2\pi$, as in (16.113). *Q.E.D.*

Example 16.52. Let $D \subset \mathbb{C}$ be a closed and *connected* domain. Let $a, b \in D$ be two points in D . Then

$$\oint_C \left(\frac{1}{z-a} - \frac{1}{z-b} \right) dz = \oint_C \frac{dz}{z-a} - \oint_C \frac{dz}{z-b} = 0$$

for any closed curve $C \subset \Omega = \mathbb{C} \setminus D$ lying outside the domain D . This is because, by connectivity of D , either C contains both points in its interior, in which case both integrals equal $2\pi i$, or C contains neither point, in which case both integrals are 0. According to Theorem 16.46, the integrand admits a single-valued anti-derivative on the domain Ω , even though each individual term is the derivative of a multiply-valued complex logarithm. The conclusion is that, while the individual logarithms are multiply-valued, their difference

$$F(z) = \log(z-a) - \log(z-b) \quad (16.124)$$

is a consistent, single-valued complex function on all of $\Omega = \mathbb{C} \setminus D$. The difference (16.124), in fact, an infinite number of possible values, differing by integer multiples of $2\pi i$; the ambiguity can be removed by choosing one of its value at a single point in Ω . These conclusions rest on the fact that D is connected, and are *not* valid, say, for the twice-punctured plane $\mathbb{C} \setminus \{a, b\}$.

We are sometimes interested in estimating the size of a complex integral. The basic inequality bounds it in terms of an arc length integral.

Proposition 16.53. *The modulus of the integral of the complex function f along a curve C is bounded by the integral of its modulus with respect to arc length:*

$$\left| \int_C f(z) dz \right| \leq \int_C |f(z)| ds. \quad (16.125)$$

Proof: We begin with a simple lemma on real integrals of complex functions.

[§] Actually, there are more subtle three-dimensional considerations that come into play, and even strings with zero winding number cannot be removed from the pole without cutting if they are linked in some nontrivial manner, cf. [115]. Can you think of an example?

Lemma 16.54. Let $f(t)$ be a complex-valued function depending on the real variable $a \leq t \leq b$. Then

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt. \quad (16.126)$$

Proof: If $\int_a^b f(t) dt = 0$, the inequality is trivial. Otherwise, let $\theta = \text{ph} \int_a^b f(t) dt$. Then, using Exercise 3.6.9,

$$\left| \int_a^b f(t) dt \right| = \text{Re} \left[e^{-i\theta} \int_a^b f(t) dt \right] = \int_a^b \text{Re} [e^{-i\theta} f(t)] dt \leq \int_a^b |f(t)| dt,$$

which proves the lemma. *Q.E.D.*

To prove the proposition, we write out the complex integral, and use (16.126) as follows:

$$\left| \int_C f(z) dz \right| = \left| \int_a^b f(z(t)) \frac{dz}{dt} dt \right| \leq \int_a^b |f(z(t))| \left| \frac{dz}{dt} \right| dt = \int_C |f(z)| ds,$$

since $|dz| = |\dot{z}| dt = \sqrt{\dot{x}^2 + \dot{y}^2} dt = ds$ is the arc length integral element (A.30). *Q.E.D.*

Corollary 16.55. If the curve C has length $L = \mathcal{L}(C)$, and $f(z)$ is an analytic function such that $|f(z)| \leq M$ for all points $z \in C$, then

$$\left| \int_C f(z) dz \right| \leq M L. \quad (16.127)$$

Lift and Circulation

In fluid mechanical applications, the complex integral can be assigned an important physical interpretation. As above, we consider the steady state flow of an incompressible, irrotational fluid. Let $f(z) = u(x, y) - i v(x, y)$ denote the complex velocity corresponding to the real velocity vector $\mathbf{v} = (u(x, y), v(x, y))^T$ at the point (x, y) .

As we noted in (16.108), the integral of the complex velocity $f(z)$ along a curve C can be written as a pair of real line integrals. In the present situation,

$$\int_C f(z) dz = \int_C (u - i v)(dx + i dy) = \int_C (u dx + v dy) - i \int_C (v dx - u dy). \quad (16.128)$$

According to (A.37, 42), the real part is the circulation integral

$$\int_C \mathbf{v} \cdot d\mathbf{x} = \int_C u dx + v dy, \quad (16.129)$$

while the imaginary part is minus the flux integral

$$\int_C \mathbf{v} \cdot \mathbf{n} ds = \int_C \mathbf{v} \wedge d\mathbf{x} = \int_C v dx - u dy, \quad (16.130)$$

along the curve C under the associated steady state fluid flow.

If the complex velocity admits a single-valued complex potential

$$\chi(z) = \varphi(z) - i\psi(z), \quad \text{where} \quad \chi'(z) = f(z)$$

— which is always the case if its domain of definition is simply connected — then the complex integral is independent of path, and one can use the Fundamental Theorem 16.45 to evaluate it:

$$\int_C f(z) dz = \chi(\beta) - \chi(\alpha) \quad (16.131)$$

for any curve C connecting α to β . Path independence of the complex integral reconfirms the path independence of the flux and circulation integrals for irrotational, incompressible fluid dynamics. The real part of formula (16.131) evaluates the circulation integral

$$\int_C \mathbf{v} \cdot d\mathbf{x} = \int_C \nabla \varphi \cdot d\mathbf{x} = \varphi(\beta) - \varphi(\alpha), \quad (16.132)$$

as the difference in the values of the (real) potential at the endpoints α, β of the curve C . On the other hand, the imaginary part of formula (16.131) computes the flux integral

$$\int_C \mathbf{v} \wedge d\mathbf{x} = \int_C \nabla \psi \cdot d\mathbf{x} = \psi(\beta) - \psi(\alpha), \quad (16.133)$$

as the difference in the values of the stream function at the endpoints of the curve. Thus, the stream function acts as a “flux potential” for the flow. Thus, for ideal fluid flows, flux is independent of path, and depends only upon the endpoints of the curve. In particular, if C is a closed contour,

$$\oint_C \mathbf{v} \cdot d\mathbf{x} = 0 = \oint_C \mathbf{v} \wedge d\mathbf{x}, \quad (16.134)$$

and so there is no net circulation or flux along any closed curve in this situation.

In aerodynamics, lift is the result of the circulation of the fluid (air) around the body, [14, 166]. More precisely, let $D \subset \mathbb{C}$ be a closed, bounded subset representing the cross-section of a cylindrical body, e.g., an airplane wing. The velocity vector field $\mathbf{v}$ of a steady state flow around the exterior of the body is defined on the domain $\Omega = \mathbb{C} \setminus D$. According to Blasius’ Theorem, the body will experience a net lift if and only if it has nonvanishing circulation integral $\oint_C \mathbf{v} \cdot d\mathbf{x} \neq 0$, where C is any simple closed contour encircling the body. However, if the complex velocity admits a single-valued complex potential in Ω , then (16.134) tells us that the circulation is automatically zero, and so the body cannot experience any lift!

Example 16.56. Let us investigate the role of lift in flow around an airfoil. Consider first the flow around a disk, as discussed in Examples 16.17 and 16.39. The Joukowski potential $\chi(z) = z + z^{-1}$ is a single-valued analytic function everywhere except at the origin $z = 0$. Therefore, the circulation integral (16.132) around any contour encircling the disk will vanish, and hence the disk experiences no net lift. This is more or less evident

from the Figure 16.13 graphing the streamlines of the flow; they are symmetric above and below the disk, and hence there cannot be any net force in the vertical direction.

Any conformal map will maintain single-valuedness of the complex potentials, and hence preserve the zero-circulation property. In particular, all the flows past airfoils constructed in Example 16.41 also admit single-valued potentials, and so also have zero circulation integral. Such an airplane will not fly, because its wings experience no lift! Of course, physical airplanes do fly, and so there must be some physical assumption we are neglecting in our treatment of flow past a body. Abandoning incompressibility or irrotationality would draw us outside the manicured gardens of complex variable theory, and into the jungles of fully nonlinear partial differential equations of fluid mechanics. Moreover, although air is slightly compressible, water is, for all practical purposes, incompressible, and hydrofoils do experience lift when traveling through water.

The only way to introduce lift into the picture is through a (single-valued) complex velocity with a non-zero circulation integral, and this requires that its complex potential be multiply-valued. The one function that we know that has such a property is the complex logarithm

$$\lambda(z) = \log(az + b), \quad \text{whose derivative} \quad \lambda'(z) = \frac{1}{az + b}$$

is single-valued away from the singularity at $z = -b/a$. Thus, we are naturally led to introduce the family of complex potentials[†]

$$\chi_k(z) = \frac{1}{2} \left(z + \frac{1}{z} \right) - ik \log z. \quad (16.135)$$

According to Exercise ■, the coefficient k must be real in order to maintain the no flux boundary conditions on the unit circle. By (16.128), the circulation is equal to the real part of the integral of the complex velocity

$$f_k(z) = \frac{d\chi_k}{dz} = \frac{1}{2} - \frac{1}{2z^2} - \frac{ik}{z}. \quad (16.136)$$

By Cauchy's Theorem 16.47 coupled with formula (16.123), if C is a curve going once around the disk in a counter-clockwise direction, then

$$\oint_C f_k(z) dz = \oint_C \left(\frac{1}{2} - \frac{1}{2z^2} - \frac{ik}{z} \right) dz = 2\pi k.$$

Therefore, when $\text{Re } k \neq 0$, the circulation integral is non-zero, and the cylinder experiences a net lift. In Figure lift■, the streamlines for the flow corresponding to a few representative values of k are plotted. Note the asymmetry of the streamlines that accounts for the lift experienced by the disk.

[†] We center the logarithmic singularity at the origin in order to maintain the no flux boundary conditions on the unit circle. Moreover, Example 16.52 tells us that more than one logarithm in the potential is redundant, since the difference of any two logarithms is effectively a single-valued function, and hence contributes nothing to the circulation integral.

When we compose the modified lift potentials (16.135) with the Joukowski transformation (16.100), we obtain a complex potential

$$\Theta_k(\zeta) = \chi_k(z) \quad \text{when} \quad \zeta = \frac{1}{2} \left(w + \frac{1}{w} \right) = \frac{1}{2} \left(a z + \beta + \frac{1}{a z + \beta} \right)$$

for flow around the corresponding airfoil — the image of the unit disk. The conformal mapping does not affect the value of the complex integrals, and hence, for any $k \neq 0$, there is a nonzero circulation around the airfoil under the modified fluid flow. This circulation is the cause of a net lift on the airfoil, and at last our airplane will fly!

However, there is now a slight embarrassment of riches, since we have designed flows around the airfoil with an *arbitrary* value $2\pi k$ for the circulation integral, and hence having an *arbitrary* amount of lift! Which of these possible flows most closely realizes the true physical version with the correct amount of lift? In his 1902 thesis, Martin Kutta hypothesized that Nature chooses the constant k so as to keep the velocity of the flow at the trailing edge of the airfoil, $\zeta = 1$, to be finite. With some additional analysis, it turns out that this condition serves to uniquely specify k , and yields a reasonably good physical approximation to the actual lift experienced by such an airfoil in flight, provided the tilt or attack angle of the airfoil in the flow is not too large. Further details, can be found in several references, including [14, 126, 116, 166].

16.6. Cauchy's Integral Formulae and the Calculus of Residues.

Cauchy's Integral Theorem 16.47 and its consequences underlie almost all applications of complex integration. The fact that we can move the contours of complex integrals around freely — as long as we do not cross over singularities of the integrand — grants us great flexibility in their evaluation. A key consequence of Cauchy's Theorem is that the value of a complex integral around a closed contour depends only upon the nature of the singularities of the integrand that happen to lie inside the contour. This observation inspires us to develop a direct method, known as the “calculus of residues”, for evaluating such integrals. The residue method effectively bypasses the Fundamental Theorem of Calculus — no antiderivatives are required! Remarkably, the method of residues can even be applied to evaluate certain types of real definite integrals, as the final examples in this section shall demonstrate.

Cauchy's Integral Formula

The first important consequence of Cauchy's Theorem is the justly famous *Cauchy integral formulae*. It enables us to compute the value of an analytic function at a point by evaluating a contour integral around a closed curve encircling the point.

Theorem 16.57. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with boundary $\partial\Omega$, and let $a \in \Omega$. If $f(z)$ is analytic on Ω , then*

$$f(a) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(z)}{z - a} dz. \quad (16.137)$$

Remark: As always, we traverse the boundary curve $\partial\Omega$ so that the domain Ω lies on our left. In most applications, Ω is simply connected, and so $\partial\Omega$ is a simple closed curve oriented in the counter-clockwise direction.

It is worth emphasizing that Cauchy's formula (16.137) is *not* a form of the Fundamental Theorem of Calculus, since we are reconstructing the function by *integration* — not its anti-derivative! Cauchy's formula is a cornerstone of complex analysis, and has no real counterpart, once again underscoring the profound difference between complex and real analysis.

Proof: We first prove that the difference quotient

$$g(z) = \frac{f(z) - f(a)}{z - a}$$

is an analytic function on all of Ω . The only problematic point is at $z = a$ where the denominator vanishes. First, by the definition of complex derivative,

$$g(a) = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} = f'(a)$$

exists and therefore $g(z)$ is well-defined and, in fact, continuous at $z = a$. Secondly, we can compute its derivative at $z = a$ directly from the definition:

$$g'(a) = \lim_{z \rightarrow a} \frac{g(z) - g(a)}{z - a} = \lim_{z \rightarrow a} \frac{f(z) - f(a) - f'(a)(z - a)}{(z - a)^2} = \frac{1}{2} f''(a),$$

where we use Taylor's Theorem C.1 (or l'Hôpital's rule) to evaluate the final limit. Knowing that g is differentiable at $z = a$ suffices to establish that it is analytic on all of Ω . Thus, we may appeal to Cauchy's Theorem 16.47, and conclude that

$$\begin{aligned} 0 &= \oint_{\partial\Omega} g(z) dz = \oint_{\partial\Omega} \frac{f(z) - f(a)}{z - a} dz = \oint_{\partial\Omega} \frac{f(z)}{z - a} dz - f(a) \oint_{\partial\Omega} \frac{dz}{z - a} \\ &= \oint_{\partial\Omega} \frac{f(z)}{z - a} dz - 2\pi i f(a). \end{aligned}$$

The second integral was evaluated using (16.123). Rearranging terms completes the proof of the Cauchy formula. *Q.E.D.*

Remark: The proof shows that if, in contrast, $a \notin \overline{\Omega}$, then the Cauchy integral vanishes:

$$\frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(z)}{z - a} dz = 0.$$

If $a \in \partial\Omega$, then the integral does not converge.

Let us see how we can apply this result to evaluate seemingly intractable complex integrals.

Example 16.58. Suppose that you are asked to compute the contour integral

$$\oint_C \frac{e^z dz}{z^2 - 2z - 3}$$

where C is a circle of radius 2 centered at the origin. A direct evaluation is not possible, since the integrand does not have an elementary anti-derivative[†]. However, we note that

$$\frac{e^z}{z^2 - 2z - 3} = \frac{e^z}{(z+1)(z-3)} = \frac{f(z)}{z+1} \quad \text{where} \quad f(z) = \frac{e^z}{z-3}$$

is analytic in the disk $|z| \leq 2$ since its only singularity, at $z = 3$, lies outside the contour C . Therefore, by Cauchy's formula (16.137), we immediately obtain the integral

$$\oint_C \frac{e^z dz}{z^2 - 2z - 3} = \oint_C \frac{f(z)}{z+1} dz = 2\pi i f(-1) = -\frac{\pi i}{2e}.$$

Note: Path independence implies that the integral has the same value on any other simple closed contour, provided it is oriented in the usual counter-clockwise direction and encircles the point $z = -1$ but not the point $z = 3$.

In this example, if the contour encloses both singularities, at $z = -1$ and 3 , then we cannot apply Cauchy's formula directly. However, as we will see, Theorem 16.57 can be adapted in a direct manner to such situations. This more general result will lead us directly to the calculus of residues, to be discussed shortly.

Derivatives by Integration

The fact that we can recover values of complex functions by integration is noteworthy. Even more amazing[†] is the fact that we can compute *derivatives* of complex functions by *integration* — turning the Fundamental Theorem on its head! Let us differentiate both sides of Cauchy's formula (16.137) with respect to a . The integrand in the Cauchy formula is sufficiently nice so as to allow us to bring the derivative inside the integral sign. Moreover, the derivative of the Cauchy integrand with respect to a is easily found:

$$\frac{\partial}{\partial a} \left(\frac{f(z)}{z-a} \right) = \frac{f(z)}{(z-a)^2}.$$

In this manner, we deduce an integral formulae for the *derivative* of an analytic function:

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz, \quad (16.138)$$

[†] At least not one listed in any integration tables, e.g., [83]. A more profound analysis, [Int], confirms that its anti-derivative *cannot* be expressed in closed form using elementary functions.

[†] Readers who have successfully tackled Exercise ■ may be less shocked by this fact.

where, as before, C is any closed curve that goes once around the point $z = a$ in a counter-clockwise direction[‡]. Further differentiation yields the general integral formulae

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad (16.139)$$

that expresses the n^{th} order derivative of a complex function in terms of a contour integral.

These remarkable formulae, which again have no counterpart in real function theory, can be used to prove our earlier claim that an analytic function is infinitely differentiable, and thereby complete the proof of Theorem 16.9.

Example 16.59. Let us compute the integral

$$\oint_C \frac{e^z dz}{z^3 - z^2 - 5z - 3} = \oint_C \frac{e^z dz}{(z+1)^2(z-3)},$$

around the circle of radius 2 centered at the origin. We use (16.138) with

$$f(z) = \frac{e^z}{z-3}, \quad \text{whereby} \quad f'(z) = \frac{(z-4)e^z}{(z-3)^2}.$$

Since $f(z)$ is analytic inside C , the integral formula (16.138) that

$$\oint_C \frac{e^z dz}{z^3 - z^2 - 5z - 3} = \oint_C \frac{f(z)}{(z+1)^2} dz = 2\pi i f'(-1) = -\frac{5\pi i}{8e}.$$

One application is the following remarkable result due to Liouville, whom we already met in Section 11.5. It says that the only bounded complex functions are the constants!

Theorem 16.60. *If $f(z)$ is analytic at all $z \in \mathbb{C}$, and satisfies $|f(z)| \leq M$ for some fixed positive number $M > 0$, then $f(z) \equiv c$ is constant.*

Proof: According to Cauchy's formula (16.137), for any point $a \in \mathbb{C}$,

$$f'(a) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{(z-a)^2} dz,$$

where we take $C_R = \{z \mid |z-a| = R\}$ to be a circle of radius R centered at $z = a$. We then estimate the complex integral using (16.125), whence

$$|f'(a)| = \frac{1}{2\pi} \left| \oint_{C_R} \frac{f(z)}{(z-a)^2} dz \right| \leq \frac{1}{2\pi} \oint_{C_R} \frac{|f(z)|}{|z-a|^2} ds \leq \frac{1}{2\pi} \oint_{C_R} \frac{M}{R^2} ds = \frac{M}{R},$$

since the length of C_R is $2\pi R$. Since $f(z)$ is analytic everywhere, we can let $R \rightarrow \infty$ and conclude that $f'(a) = 0$. Since this occurs for all possible points a , we conclude that $f'(z) \equiv 0$ is everywhere zero, which suffices to prove constancy of $f(z)$. *Q.E.D.*

[‡] Or, more generally, has winding number +1 around the point $z = a$.

One outstanding application of Liouville's Theorem 16.60 is a proof of the Fundamental Theorem of Algebra, first proved by Gauss in 1799; see [71] for an extensive discussion. Although it is, in essence, a purely algebraic result, this proof relies in an essential way on complex analysis and complex integration.

Theorem 16.61. *Every nonconstant (complex or real) polynomial $f(z)$ has at least one complex root $z_0 \in \mathbb{C}$.*

Proof: Suppose

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \neq 0 \quad \text{for all } z \in \mathbb{C}. \quad (16.140)$$

Then we claim that its reciprocal

$$g(z) = \frac{1}{f(z)} = \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

satisfies the hypotheses of Theorem 16.60, and hence must be constant, in contradiction to our hypothesis. Therefore, $f(z)$ cannot be non-zero for all z , and this establishes the result.

To prove the claim, note first that our nonvanishing assumption (16.140) implies that $g(z)$ is analytic for all $z \in \mathbb{C}$. Moreover,

$$\begin{aligned} |f(z)| &= |z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \\ &\leq |z|^n (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|), \end{aligned} \quad \text{whenever } |z| \geq 1,$$

which implies that $|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$, and so

$$|g(z)| = \frac{1}{|f(z)|} \longrightarrow 0 \quad \text{as } |z| \rightarrow \infty.$$

This suffices to prove that $|g(z)| \leq M$ is bounded for $z \in \mathbb{C}$. *Q.E.D.*

Corollary 16.62. *Every complex polynomial of degree n can be factored,*

$$f(z) = a_n (z - z_1) (z - z_2) \cdots (z - z_n)$$

where $z_1, \dots, z_n$ are the roots of $f(z)$, listed in accordance with their multiplicity.

Proof: Theorem 16.61 guarantees that there is at least one point $z_1 \in \mathbb{C}$ where $f(z_1) = 0$. Therefore, by the rules of polynomial factorization, we can write

$$f(z) = (z - z_1) g(z)$$

where $g(z)$ is a polynomial of degree $n - 1$. A straightforward induction on the degree of the polynomial completes the proof. *Q.E.D.*

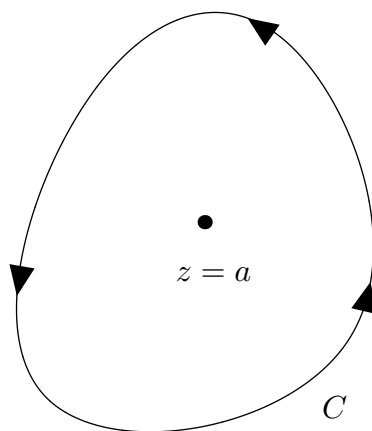


Figure 16.30. Computing a Residue.

The Calculus of Residues

Cauchy's Theorem and Integral Formulae provide us with some amazingly versatile tools for evaluating complicated complex integrals. The upshot is that one only needs to understand the singularities of the integrand within the domain of integration — no indefinite integration is required! With a little more work, we are led to a general method for efficiently computing contour integrals, known as the *Calculus of Residues*. While the residue method has no counterpart in real integration theory, it can, remarkably, be used to evaluate a large variety of interesting definite real integrals, including many without an explicitly known anti-derivative.

Definition 16.63. Let $f(z)$ be an analytic function for all z near, but not equal to a . The *residue* of $f(z)$ at the point $z = a$ is defined by the contour integral

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{2\pi i} \oint_C f(z) dz, \quad (16.141)$$

where C is any simple, closed curve that contains a in its interior, oriented, as always, in a counter-clockwise direction, and such that $f(z)$ is analytic everywhere inside C except at the point $z = a$; see Figure 16.30. For example, C could be a small circle centered at a . The residue is a complex number, and tells us important information about the singularity of $f(z)$ at $z = a$.

The simplest example is the monomial function $f(z) = cz^n$, where c is a complex constant and the exponent n is assumed to be an integer. (Residues are not defined at branch points.) According to (16.112),

$$\operatorname{Res}_{z=0} cz^n = \frac{1}{2\pi i} \oint_C cz^n dz = \begin{cases} 0, & n \neq -1, \\ c, & n = -1. \end{cases} \quad (16.142)$$

Thus, only the case $n = -1$ gives a nonzero residue. The residue singles out the function $1/z$, which, not coincidentally, is the only one with a logarithmic, and multiply-valued, antiderivative.

Cauchy's Theorem 16.47, when applied to the integral in (16.141), implies that if $f(z)$ is analytic at $z = a$, then it has zero residue at a . Therefore, all the monomials, including $1/z$, have zero residue at any nonzero point:

$$\operatorname{Res}_{z=a} c z^n = 0 \quad \text{for} \quad a \neq 0. \quad (16.143)$$

Since integration is a linear operation, the residue is a linear operator, mapping complex functions to complex numbers:

$$\operatorname{Res}_{z=a} [f(z) + g(z)] = \operatorname{Res}_{z=a} f(z) + \operatorname{Res}_{z=a} g(z), \quad \operatorname{Res}_{z=a} [c f(z)] = c \operatorname{Res}_{z=a} f(z), \quad (16.144)$$

for any complex constant c . Thus, by linearity, the residue of any finite linear combination of monomials,

$$f(z) = \frac{c_{-m}}{z^m} + \frac{c_{-m+1}}{z^{m-1}} + \cdots + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots + c_n z^n = \sum_{k=-m}^n c_k z^k,$$

is equal to

$$\operatorname{Res}_{z=0} f(z) = c_{-1}.$$

Thus, the residue effectively picks out the coefficient of the term $1/z$ in such an expansion.

The easiest nontrivial residues to compute are at the poles of a function. According to (16.29), the function $f(z)$ has a *simple pole* at $z = a$ if

$$h(z) = (z - a) f(z) \quad (16.145)$$

is analytic at $z = a$ and $h(a) \neq 0$. The next result allows us to bypass the contour integral when evaluating such a residue.

Lemma 16.64. *If $f(z) = \frac{h(z)}{z - a}$ has a simple pole at $z = a$, then $\operatorname{Res}_{z=a} f(z) = h(a)$.*

Proof: We substitute the formula for $f(z)$ into the definition (16.141), and so

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{2\pi i} \oint_C f(z) dz = \frac{1}{2\pi i} \oint_C \frac{h(z) dz}{z - a} = h(a)$$

by Cauchy's formula (16.137).

Q.E.D.

Example 16.65. Consider the function

$$f(z) = \frac{e^z}{z^2 - 2z - 3} = \frac{e^z}{(z + 1)(z - 3)}.$$

From the factorization of the denominator, we see that $f(z)$ has simple pole singularities at $z = -1$ and $z = 3$. The residues are given, respectively, by

$$\operatorname{Res}_{z=-1} \frac{e^z}{z^2 - 2z - 3} = \frac{e^z}{z - 3} \Big|_{z=-1} = -\frac{1}{4e}, \quad \operatorname{Res}_{z=3} \frac{e^z}{z^2 - 2z - 3} = \frac{e^z}{z + 1} \Big|_{z=3} = \frac{e^3}{4}.$$

Since $f(z)$ is analytic everywhere else, its residue at any other point is automatically 0.

Recall that a function $g(z)$ is said to have *simple zero* at $z = a$ provided

$$g(z) = (z - a) k(z)$$

where $k(z)$ is analytic at $z = a$ and $k(a) = g'(a) \neq 0$. If $f(z)$ is analytic at $z = a$, then the quotient

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z - a) k(z)}$$

has a simple pole at $z = a$, with residue

$$\operatorname{Res}_{z=a} \frac{f(z)}{g(z)} = \operatorname{Res}_{z=a} \frac{f(z)}{(z - a) k(z)} = \frac{f(a)}{k(a)} = \frac{f(a)}{g'(a)} \quad (16.146)$$

by Lemma 16.64. More generally, if $z = a$ is a zero of order n of

$$g(z) = (z - a)^n k(z), \quad \text{so that} \quad k(a) = \frac{g^{(n)}(a)}{n!} \neq 0,$$

then

$$\operatorname{Res}_{z=a} \frac{f(z)}{g(z)} = \frac{1}{(n-1)!} \left. \frac{d^{n-1}}{dz^{n-1}} \left(\frac{f(z)}{k(z)} \right) \right|_{z=a}. \quad (16.147)$$

The proof of the latter formula is left as Exercise ■.

Example 16.66. As an illustration, let us compute the residue of $\sec z = 1/\cos z$ at the point $z = \frac{1}{2}\pi$. Note that $\cos z$ has a simple zero at $z = \frac{1}{2}\pi$ since its derivative, namely $-\sin z$, is nonzero there. Thus, according to (16.146) with $f(z) \equiv 1$,

$$\operatorname{Res}_{z=\pi/2} \sec z = \operatorname{Res}_{z=\pi/2} \frac{1}{\cos z} = \frac{-1}{\sin \frac{1}{2}\pi} = -1.$$

The direct computation of the residue using the defining contour integral (16.141) is much harder.

The Residue Theorem

Residues are the building blocks of a general method for computing contour integrals of analytic functions. The *Residue Theorem* says that the value of the integral of a complex function around a closed curve depends only on its residues at the enclosed singularities. Since the residues can be computed directly from the function, the resulting formula provides an effective mechanism for painless evaluation of complex integrals, that completely avoids the anti-derivative. Indeed, the residue method continues to be effective even when the integrand does not have an anti-derivative that can be expressed in terms of elementary functions.

Theorem 16.67. Let C be a simple, closed curve, oriented in the counter-clockwise direction. Suppose $f(z)$ is analytic everywhere inside C except at a finite number of singularities, $a_1, \dots, a_n$. Then

$$\frac{1}{2\pi i} \oint_C f(z) dz = \operatorname{Res}_{z=a_1} f(z) + \dots + \operatorname{Res}_{z=a_n} f(z). \quad (16.148)$$

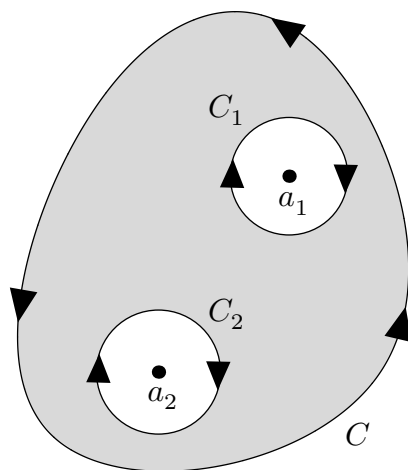


Figure 16.31. Residues Inside a Contour.

Keep in mind that only the singularities that lie *inside* the contour C contribute to the residue formula (16.148).

Proof: We draw a small circle C_i around each singularity a_i . We assume the circles all lie inside the contour C and do not cross each other, so that a_i is the only singularity contained within C_i ; see Figure 16.31. Definition 16.63 implies that

$$\operatorname{Res}_{z=a_i} f(z) = \frac{1}{2\pi i} \oint_{C_i} f(z) dz, \quad (16.149)$$

where the line integral is taken in the counter-clockwise direction around C_i .

Consider the domain Ω consisting of all points z which lie inside the given curve C , but outside all the small circles $C_1, \dots, C_n$; this is the shaded region in Figure 16.31. By our construction, the function $f(z)$ is analytic on Ω , and hence by Cauchy's Theorem 16.47, the integral of $f(z)$ around the boundary $\partial\Omega$ is zero. The boundary $\partial\Omega$ must be oriented consistently, so that the domain is always lying on one's left hand side. This means that the outside contour C should be traversed in a counter-clockwise direction, whereas the inside circles C_i are taken in a *clockwise* direction. Therefore, the integral around the boundary of the domain Ω can be broken up into a difference

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_{\partial\Omega} f(z) dz = \frac{1}{2\pi i} \oint_C f(z) dz - \sum_{i=1}^n \frac{1}{2\pi i} \oint_{C_i} f(z) dz \\ &= \frac{1}{2\pi i} \oint_C f(z) dz - \sum_{i=1}^n \operatorname{Res}_{z=a_i} f(z) dz. \end{aligned}$$

The minus sign converts the circular integrals to the counterclockwise orientation used in the definition (16.149) of the residues. Rearranging the final identity leads to the residue formula (16.148). Q.E.D.

Example 16.68. Let us use residues to evaluate the contour integral

$$\oint_{C_r} \frac{e^z}{z^2 - 2z - 3} dz$$

where C_r denotes the circle of radius r centered at the origin. According to Example 16.65, the integrand has two singularities at -1 and 3 , with respective residues $-1/(4e)$ and $e^3/4$. If the radius of the circle is $r > 3$, then it goes around both singularities, and hence by the residue formula (16.148)

$$\oint_C \frac{e^z dz}{z^2 - 2z - 3} = 2\pi i \left(-\frac{1}{4e} + \frac{e^3}{4} \right) = \frac{(e^4 - 1)\pi i}{2e}, \quad r > 3.$$

If the circle has radius $1 < r < 3$, then it only encircles the singularity at -1 , and hence

$$\oint_C \frac{e^z}{z^2 - 2z - 3} dz = -\frac{\pi i}{2e}, \quad 1 < r < 3.$$

If $0 < r < 1$, the function has no singularities inside the circle and hence, by Cauchy's Theorem 16.47, the integral is 0. Finally, when $r = 1$ or $r = 3$, the contour passes through a singularity, and the integral does not converge.

Evaluation of Real Integrals

One important — and unexpected — application of the Residue Theorem 16.67 is to aid in the evaluation of certain definite real integrals. Of particular note is that it even applies to cases in which one is unable to evaluate the corresponding indefinite integral in closed form. Nevertheless, converting the definite real integral into (part of a) complex contour integral leads to a direct evaluation via the calculus of residues that sidesteps the difficulties in finding the antiderivative.

The method treats two basic types of real integral, although numerous variations appear in more extensive treatments of the subject. The first category are real trigonometric integrals of the form

$$I = \int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta. \quad (16.150)$$

Such integrals can often be evaluated by converting them into complex integrals around the unit circle $C = \{ |z| = 1 \}$. If we set

$$z = e^{i\theta} \quad \text{so} \quad \frac{1}{z} = e^{-i\theta},$$

then

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2} \left(z + \frac{1}{z} \right), \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i} \left(z - \frac{1}{z} \right). \quad (16.151)$$

Moreover,

$$dz = de^{i\theta} = ie^{i\theta} d\theta = iz d\theta, \quad \text{and so} \quad d\theta = \frac{dz}{iz}. \quad (16.152)$$

Therefore, the integral (16.150) can be written in the complex form

$$I = \oint_C F\left(\frac{z+z^{-1}}{2}, \frac{z+z^{-1}}{2i}\right) \frac{dz}{iz}. \quad (16.153)$$

If we know that the resulting complex integrand is well-defined and single-valued, except, possibly, for a finite number of singularities inside the unit circle, then the residue formula (16.148) tells us that the integral can be directly evaluated by adding together its residues and multiplying by $2\pi i$.

Example 16.69. We compute the relatively simple example

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}.$$

Applying the substitution (16.153), we find

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \oint_C \frac{dz}{iz \left[2 + \frac{1}{2}(z + z^{-1})\right]} = -i \oint_C \frac{2 dz}{z^2 + 4z + 1}.$$

The complex integrand has singularities where its denominator vanishes:

$$z^2 + 4z + 1 = 0, \quad \text{so that} \quad z = -2 \pm \sqrt{3}.$$

Only one of these singularities, namely $-2 + \sqrt{3}$ lies inside the unit circle. Therefore, applying (16.146), we find

$$-i \oint_C \frac{2 dz}{z^2 + 4z + 1} = 2\pi \operatorname{Res}_{z=-2+\sqrt{3}} \frac{2}{z^2 + 4z + 1} = \frac{4\pi}{2z + 4} \Big|_{z=-2+\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

As you may recall from first year calculus, this particular integral can, in fact, be computed directly via a trigonometric substitution. However, the integration is not particularly pleasant, and, with a little practice, the residue method is seen to be an easier method. Moreover, it straightforwardly applies to situations where no elementary anti-derivative exists.

Example 16.70. The goal is to evaluate the definite integral

$$\int_0^\pi \frac{\cos 2\theta}{3 - \cos \theta} d\theta.$$

The first thing to note is that the integral omny runs from 0 to π and so is not explicitly of the form (16.150). However, note that the integrand is even, and so

$$\int_0^\pi \frac{\cos 2\theta}{3 - \cos \theta} d\theta = \frac{1}{2} \int_{-\pi}^\pi \frac{\cos 2\theta}{3 - \cos \theta} d\theta,$$

which will turn into a contour integral around the entire unit circle under the substitution (16.151). Also note that

$$\cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{1}{2} \left(z^2 + \frac{1}{z^2} \right),$$

and so

$$\int_{-\pi}^{\pi} \frac{\cos 2\theta}{3 - \cos \theta} d\theta = \oint_C \frac{\frac{1}{2}(z^2 + z^{-2})}{3 - \frac{1}{2}(z + z^{-1})} \frac{dz}{iz} = i \oint_C \frac{z^4 + 1}{z^2(z^2 - 6z + 1)} dz.$$

The denominator has 4 roots — at 0, $3 - 2\sqrt{2}$, and $3 + 2\sqrt{2}$ — but the last one does not lie inside the unit circle and so can be ignored. We use (16.146) with $f(z) = (z^4 + 1)/z^2$ and $g(z) = z^2 - 6z + 1$ to compute

$$\operatorname{Res}_{z=3-2\sqrt{2}} \frac{z^4 + 1}{z^2(z^2 - 6z + 1)} = \frac{z^4 + 1}{z^2} \bigg|_{z=3-2\sqrt{2}} \frac{1}{2z - 6} = \frac{17}{4} \sqrt{2},$$

whereas (16.147) is used to compute

$$\operatorname{Res}_{z=0} \frac{z^4 + 1}{z^2(z^2 - 6z + 1)} = \frac{d}{dz} \left(\frac{z^4 + 1}{z^2 - 6z + 1} \right) \bigg|_{z=0} = \frac{2(z^5 - 9z^4 + 2z^3 - z + 3)}{(z^2 - 6z + 1)^2} \bigg|_{z=0} = 6.$$

Therefore,

$$\int_0^{\pi} \frac{\cos 2\theta}{3 - \cos \theta} d\theta = \pi \left[\operatorname{Res}_{z=0} \frac{z^4 + 1}{z^2(z^2 - 6z + 1)} + \operatorname{Res}_{z=3-2\sqrt{2}} \frac{z^4 + 1}{z^2(z^2 - 6z + 1)} \right] = -6\pi + \frac{17}{4} \sqrt{2} \pi.$$

A second type of real integral that can often be evaluated by complex residues are integrals over the entire real line, from $-\infty$ to ∞ . Here the technique is a little more subtle, and we sneak up on the integral by using larger and larger closed contours that include more and more of the real axis. The basic idea is contained in the following example.

Example 16.71. The problem is to evaluate the real integral

$$I = \int_0^{\infty} \frac{\cos x}{1 + x^2} dx. \quad (16.154)$$

The corresponding indefinite integral cannot be evaluated in elementary terms, and so we are forced to rely on the calculus of residues. We begin by noting that the integrand is even, and hence the integral $I = \frac{1}{2} J$ is one half the integral

$$J = \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} dx$$

over the entire real line. Moreover, for x real, we can write

$$\frac{\cos x}{1 + x^2} = \operatorname{Re} \frac{e^{ix}}{1 + x^2}, \quad \text{and hence} \quad J = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} dx. \quad (16.155)$$

Let C_R be the closed contour consisting of a large semicircle of radius $R \gg 0$, which we denote by S_R , connected at its ends by the real interval $-R \leq x \leq R$, as plotted

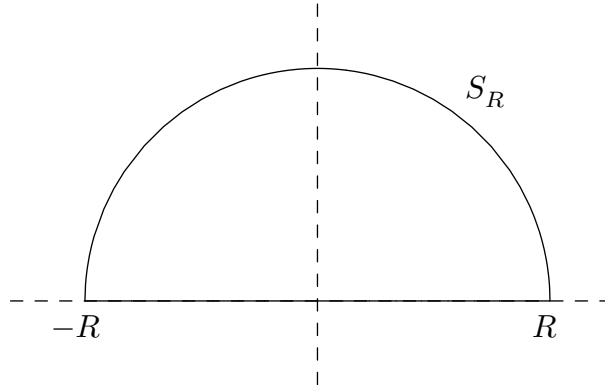


Figure 16.32. The Semicircular Contour.

in Figure 16.32, and having the usual counterclockwise orientation. The corresponding contour integral

$$\oint_{C_R} \frac{e^{iz} dz}{1+z^2} = \int_{-R}^R \frac{e^{ix} dx}{1+x^2} + \int_{S_R} \frac{e^{iz} dz}{1+z^2} \quad (16.156)$$

breaks up into two pieces: the first over the real interval, and the second over the semicircle. As the radius $R \rightarrow \infty$, the semicircular contour C_R includes more and more of the real axis, and so the first integral gets closer and closer to our desired integral (16.155). If we can prove that the second, semicircular integral goes to zero, then we will be able to evaluate the integral over the real axis by contour integration, and hence by the method of residues. Our hope that the semicircular integral is small seems reasonable, since the integrand $(1+z^2)^{-1}e^{iz}$ gets smaller and smaller as $|z| \rightarrow \infty$ *provided* $\text{Im } z \geq 0$. (Why?) A rigorous verification of this fact will appear at the end of the example.

According to the Residue Theorem 16.67, the integral (16.156) is equal to the sum of all the residues at the singularities of $f(z)$ lying inside the contour C_R . Now e^z is analytic everywhere, and so the singularities occur where the denominator vanishes, i.e., $z^2 = -1$, and so are at $z = \pm i$. Since the semicircle lies in the upper half plane $\text{Im } z > 0$, only the singularity $z = +i$ lies inside — and then only when $R > 1$. To compute the residue, we use (16.146) to evaluate

$$\text{Res}_{z=i} \frac{e^{iz}}{1+z^2} = \left. \frac{e^{iz}}{2z} \right|_{z=i} = \frac{e^{-1}}{2i} = \frac{1}{2ie}.$$

Therefore, by (16.148),

$$\frac{1}{2\pi i} \oint_{C_R} \frac{e^{iz} dz}{1+z^2} = \frac{1}{2ie}, \quad \text{provided } R > 1.$$

Thus, assuming the semicircular part of the integral does indeed become vanishingly small as $R \rightarrow \infty$, we conclude that

$$\int_{-\infty}^{\infty} \frac{e^{ix} dx}{1+x^2} = \lim_{R \rightarrow \infty} \oint_{C_R} \frac{e^{iz} dz}{1+z^2} = 2\pi i \frac{1}{2ie} = \frac{\pi}{e}.$$

Incidentally, the integral is real because its imaginary part,

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{1+x^2} = 0,$$

is the integral of an odd function which is automatically zero. Consequently,

$$I = \int_0^{\infty} \frac{\cos x \, dx}{1+x^2} = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{ix} \, dx}{1+x^2} = \frac{\pi}{2e}, \quad (16.157)$$

which is the desired result.

To complete the argument, let us estimate the size of the semicircular integral. The integrand is bounded by

$$\left| \frac{e^{iz}}{1+z^2} \right| \leq \frac{1}{1+|z|^2} = \frac{1}{1+R^2} \quad \text{whenever} \quad |z| = R, \quad \operatorname{Im} z \geq 0,$$

where we are using the fact that

$$|e^{iz}| = e^{-y} \leq 1 \quad \text{whenever} \quad z = x + iy \quad \text{with} \quad y \geq 0.$$

According to Corollary 16.55, the size of the integral of a complex function is bounded by its maximum modulus along the curve times the length of the semicircle, namely πR . Thus, in our case,

$$\left| \int_{S_R} \frac{e^{iz} \, dz}{1+z^2} \right| \leq \frac{\pi R}{1+R^2} \leq \frac{\pi}{R} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty,$$

as required.

Example 16.72. Here we will use residues to evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}. \quad (16.158)$$

The indefinite integral can, in fact, be found by the method of partial fractions, but, as you may know, this is not a particularly pleasant task. Let us try the method of residues. Let C_R denote the same semicircular contour as in Figure 16.32. The integrand has pole singularities where the denominator vanishes, i.e., $z^4 = -1$, and so at the four fourth roots of -1 . These are

$$e^{\pi i/4} = \frac{1+i}{\sqrt{2}}, \quad e^{3\pi i/4} = \frac{-1+i}{\sqrt{2}}, \quad e^{5\pi i/4} = \frac{1-i}{\sqrt{2}}, \quad e^{7\pi i/4} = \frac{-1-i}{\sqrt{2}}.$$

Only the first two roots lie inside C_R when $R > 1$. Their residues can be computed using (16.146):

$$\begin{aligned} \operatorname{Res}_{z=e^{\pi i/4}} \frac{1}{1+z^4} &= \frac{1}{4z^3} \bigg|_{z=e^{\pi i/4}} = \frac{e^{-3\pi i/4}}{4} = \frac{-1-i}{4\sqrt{2}}, \\ \operatorname{Res}_{z=e^{3\pi i/4}} \frac{1}{1+z^4} &= \frac{1}{4z^3} \bigg|_{z=e^{3\pi i/4}} = \frac{e^{-9\pi i/4}}{4} = \frac{1-i}{4\sqrt{2}}. \end{aligned}$$

Therefore, by the residue formula (16.148),

$$\oint_{C_R} \frac{dz}{1+z^4} = 2\pi i \left(\frac{-1-i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}}. \quad (16.159)$$

On the other hand, we can break up the contour integral into an integral along the real axis and an integral around the semicircle:

$$\oint_{C_R} \frac{dz}{1+z^4} = \int_{-R}^R \frac{dx}{1+x^4} + \int_{S_R} \frac{dz}{1+z^4}.$$

The first integral goes to the desired real integral as the radius $R \rightarrow \infty$. On the other hand, on a large semicircle $|z| = R$, the integrand is small:

$$\left| \frac{1}{1+z^4} \right| \leq \frac{1}{1+|z|^4} = \frac{1}{1+R^4} \quad \text{when} \quad |z| = R.$$

Thus, using Corollary 16.55, the integral around the semicircle can be bounded by

$$\left| \int_{S_R} \frac{dz}{1+z^4} \right| \leq \frac{1}{1+R^4} \pi R \leq \frac{\pi}{R^3} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.$$

Thus, as $R \rightarrow \infty$, the complex integral (16.159) goes to the desired real integral (16.158), and so

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

Note that the result is real and positive, as it must be.